A NOTE ON EVENNESS AND REGULARITY OF EQUIVARIANT COMPLEX BORDISM OVER ABELIAN GROUPS

MARKUS HAUSMANN

Abstract. We give a new proof that the homotopical A-equivariant complex bordism ring is concentrated in even degrees, for every abelian compact Lie group A, by constructing and utilizing additional structure maps between relative geometric fixed points.

1. Introduction

Let A be an abelian compact Lie group, and let MU_A denote tom Dieck's homotopical equivariant complex bordism spectrum [tD70]. The following was stated without proof in [Löf73]:

Theorem 1 (Evenness)**.** *For every abelian compact Lie group A, the equivariant homotopy groups $\pi_*^A \mathrm{MU}_A$ are concentrated in even degrees.*

To the best of the author's knowledge, the first published proof was given in [Com96], by an induction on the rank of A and isotropy separation techniques. (The case of the trivial group is of course older and due to Milnor [Mil60].) Evenness is a rare and very useful property of spectra, further emphasized by the recent work [HRW26] on the even filtration. Evenness of $\pi_*^A \mathrm{MU}_A$ specifically is used in work in progress of Devalapurkar–Hahn–Raksit–Yuan on prismatization of E_∞-rings, see [Hah26, Sec. 3], and also entered as a prerequisite in the author's identification of $\pi_*^A \mathrm{MU}_A$ with the A-equivariant Lazard ring [Hau22].

Since MU_A is canonically complex oriented, it comes with a theory of Euler classes for complex vector bundles. In particular, every character $V \in A^*$ has an associated Euler class e_V in the MU_A-cohomology of a point, i.e., the coefficient ring $\pi_*^A \mathrm{MU}_A$. In [Sin01, Thm. 5.1], Sinha showed that evenness (Theorem 1) implies the following:

Theorem 2 (Regularity)**.** *For every abelian compact Lie group A and every character $V \in A^*$ of infinite order, the Euler class e_V is a regular element in the ring $\pi_*^A \mathrm{MU}_A$.*

The case of A a torus was shown earlier in [Löf74]. Theorems 1 and 2 are closely related. In fact, through a variation of Sinha's argument it is not hard to see that for a complex oriented Ab-global ring spectrum E in the sense of Schwede [Sch18], such as global equivariant complex bordism, evenness of all $\pi_*^A E_A$ is equivalent to regularity of Euler classes for characters of infinite order plus evenness of all geometric fixed points $\Phi_*^A E_A$ (see Lemma 2.3).

In this note we turn the previous line of argument around and give a short, direct proof of regularity (Theorem 2). Since evenness of $\Phi_*^A \mathrm{MU}_A$ is a consequence of the evenness of non-equivariant MU and general properties of geometric fixed points of equivariant Thom spectra, this yields a new proof of the evenness of $\pi_*^A \mathrm{MU}_A$ (Theorem 1). Our main ingredient is the observation that the restriction maps $\pi_*^A \mathrm{MU}_A \to \pi_*^B \mathrm{MU}_B$ admit multiplicative sections after inverting suitable collections of Euler classes (Proposition 3.1).

We hope that the approach described here is useful also in other settings where the methods of [Com96] do not apply directly. One example is the work of Allen–Piessevaux, who use a motivic enhancement of this strategy to prove several new results on equivariant motivic bordism MGL_A over the complex numbers, including Chow vanishing, see [AP25, Thm. C].

The organization is as follows: In Section 2 we describe the relationship between evenness and regularity for complex oriented Ab-global ring spectra. In Section 3 we show evenness and regularity of $\pi_*^A \mathrm{MU}_A$ assuming the existence of a section of Euler-inverted restriction maps. In Section 4 we then give a quick explicit construction of these through a point-set model for complex equivariant bordism. In the appendix we provide a sketch of a more categorical spectrum level construction, using the equivariant Thom spectrum formalism for incomplete universes.

Acknowledgements: I thank Emma Brink, Tobias Lenz, Lucas Piessevaux and Stefan Schwede for several helpful discussions on this topic. This work was supported by the Hausdorff Center for Mathematics at the University of Bonn (DFG GZ 2047/1, project ID 390685813) and by the European Research Council (ERC) under Horizon Europe (Starting Grant BorSym, ID: 101163408).

2. Complex oriented global ring spectra

In this section we quickly recall the relevant notions from global homotopy theory and complex orientations, and describe the relationship between regularity and evenness.

By an Ab-*global ring spectrum* we mean a global spectrum indexed on the family Ab of abelian compact Lie groups, in the sense of [Sch18], which is further equipped with a commutative ring structure in the global homotopy category. Given $A \in \mathrm{Ab}$, we write E_A for the associated A-spectrum. Then for every group homomorphism $\alpha\colon B \to A$ the global structure induces a homotopy ring map $\alpha^* E_A \to E_B$ of B-spectra which is an equivalence if α is injective, cf. [LNP25]. We recall from [Hau22, Def. 5.18] that a *complex orientation* of an Ab-global ring spectrum E is an $RU(\mathbb{T})$-graded unit $t \in \pi^{\mathbb{T}}_{\tau-2} E_{\mathbb{T}}$, where $\mathbb{T}$ denotes the circle group and τ its tautological complex character. Moreover, t is required to restrict to 1 at the trivial group. This gives rise to Thom classes $t_V = V^* t \in \pi^A_{V-2} E_A$ for all characters of abelian compact Lie groups A. Precomposition with the inclusion $S^0 \hookrightarrow S^V$ yields the Euler class $e_V \in \pi^A_{-2} E_A$. As explained in [Hau22, Sec. 5.4.1], each E_A inherits a complex orientation in the A-equivariant sense, and these are suitably compatible.

One of the main tools for studying complex oriented equivariant ring spectra are long exact sequences relating Euler classes and restriction maps, constructed as follows. Let $A \in \mathrm{Ab}$ and $V \in A^*$ be a character of infinite order. Then there is a cofiber sequence in based A-spaces

$$A/\ker(V)_+ \to S^0 \xrightarrow{a_V} S^V,$$

using that $A/\ker(V)$ identifies with the unit sphere $S(V)$ inside V. Taking suspension spectra, mapping into E_A and applying the Thom isomorphism $\mathrm{map}(S^V, E_A) \simeq \mathrm{map}(S^2, E_A)$, one finds that the cofiber sequence induces a long exact sequence of the form

$$\cdots \to \pi^A_{*+2} E_A \xrightarrow{e_V \cdot} \pi^A_* E_A \xrightarrow{\mathrm{res}^A_{\ker(V)}} \pi^{\ker(V)}_* E_{\ker(V)} \to \dots \tag{1}$$

Lemma 2.1. *Let E be a complex oriented* Ab*-global ring spectrum. Then the following are equivalent:*

(i) For every $A \in \mathrm{Ab}$ and $V \in A^*$ of infinite order, the Euler class $e_V \in \pi^A_* E_A$ is regular.
(ii) For every $A \in \mathrm{Ab}$ and every linearly independent tuple of A-characters $(V_1, \dots, V_n)$, the sequence $(e_{V_1}, \dots, e_{V_n})$ is regular in $\pi^A_* E_A$.
(iii) For every inclusion $B \subseteq A \in \mathrm{Ab}$, the restriction map $\pi^A_* E_A \to \pi^B_* E_B$ is surjective.

Proof. This is essentially [Hau22, Lem. 5.23], but we recall the argument for the convenience of the reader. Let $A \in \mathrm{Ab}$ and $V \in A^*$ be of infinite order. Then the long exact sequence (1) implies that $e_V \in \pi_*^A E_A$ is regular if and only if the restriction $\pi_*^A E_A \to \pi_*^{\ker(V)} E_{\ker(V)}$ is surjective. Hence we find that (i) is equivalent to all restriction maps for inclusions with quotient isomorphic to $\mathbb{T}$ being surjective. We claim that the latter implies (iii), i.e., surjectivity of all restriction maps. If $B \subseteq A$ is a general inclusion, we can find $A \subseteq A'$ such that A'/B is a torus (for example we can choose A' itself to be a torus). Then the kernel of $(A')^* \to B^*$ is a finitely generated free abelian group, and hence we can find a filtration

$$B = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = A'$$

such that each $A_i \subseteq A_{i+1}$ is the kernel of a character of A_{i+1}^* of infinite order. By assumption each $\mathrm{res}_{A_i}^{A_{i+1}}$ is surjective, and hence so is the composite $\mathrm{res}_B^{A'}$. Since $\mathrm{res}_B^{A'} = \mathrm{res}_B^A \circ \mathrm{res}_A^{A'}$, it follows that res_B^A is surjective as well. This finishes the proof that (i) and (iii) are equivalent. Since (ii) clearly implies (i), it suffices to see that (i) implies (ii). This follows by induction on n, since if $(V_1, \ldots, V_n)$ is linearly independent and e_{V_1} is regular, then by (1) we have $(\pi_*^A E_A)/e_{V_1} \cong \pi_*^{\ker(V_1)} E_{\ker(V_1)}$ and the restriction of $(V_2, \ldots, V_n)$ to $\ker(V_1)^*$ is linearly independent. □

Definition 2.2 (Regularity)**.** A complex oriented global ring spectrum E is called *regular* if the three equivalent conditions of Lemma 2.1 are satisfied.

We then have the following, where $\Phi_*^A E_A$ denotes the geometric fixed point homotopy groups of E_A, which come with a comparison map $\pi_*^A E_A \to \Phi_*^A E_A$. In the presence of Euler classes this map is a localization away from all e_V with $V \in A^*$ non-trivial, i.e.,

$$\Phi_*^A E_A \cong \pi_*^A E_A[e_V^{-1} \mid \epsilon \neq V \in A^*].$$

Lemma 2.3. *A complex oriented* Ab*-global ring spectrum E is even (i.e., $\pi_*^A E_A$ is even for all $A \in \mathrm{Ab}$) if and only if it is regular and $\Phi_*^A E_A$ is even for every $A \in \mathrm{Ab}$.*

Proof. Let E be even. Then $\Phi_*^A E_A$ is even as well, since it is a localization of $\pi_*^A E_A$. Moreover, if $A \in \mathrm{Ab}$ and $V \in A^*$ has infinite order, the long exact sequence (1) implies that if $\pi_*^A E_A$ and $\pi_*^{\ker(V)} E_{\ker(V)}$ are even, then $\mathrm{res}_{\ker(V)}^A$ must be surjective and hence e_V is regular.

Now assume that E is regular and that $\Phi_*^A E_A$ is even for all $A \in \mathrm{Ab}$. Then, for every torus $\mathbb{T}^r$, the localization map $\pi_*^{\mathbb{T}^r} E_{\mathbb{T}^r} \to \Phi_*^{\mathbb{T}^r} E_{\mathbb{T}^r}$ is injective, since every non-trivial character has infinite order and hence each e_V is regular by assumption. Hence, $\pi_*^{\mathbb{T}^r} E_{\mathbb{T}^r}$ is even. A general $A \in \mathrm{Ab}$ embeds into some torus $\mathbb{T}^r$, and by the third characterization of regularity from Lemma 2.1 the resulting restriction map $\pi_*^{\mathbb{T}^r} E_{\mathbb{T}^r} \to \pi_*^A E_A$ is surjective, implying that $\pi_*^A E_A$ is even as well. □

Example 2.4 (Equivariant K-theory $\mathbf{KU}$)**.** The global complex K-theory spectrum $\mathbf{KU}$ (see [Sch18, Sec. 6.4]) is complex oriented via the Bott class, and even. In fact, $\pi_*^G \mathrm{KU}_G$ is even for all compact Lie groups G.

Example 2.5 (Borel theories)**.** Let E be a non-equivariant homotopy-commutative homotopy ring spectrum and bE its Ab-global Borel theory (see, e.g., [Sch18, Constr. 4.5.21]). A global complex orientation of the Ab-global Borel theory bE is equivalent data to a complex orientation of E, since there is a natural isomorphism

$$\pi_{\tau-2}^{\mathbb{T}}(bE)_{\mathbb{T}} \cong E^2(S^\tau \wedge_{\mathbb{T}} E\mathbb{T}_+) \cong E^2(\mathbb{C}P^\infty).$$

Then $\pi_*^A(bE)_A \cong E^{-*}(BA_+)$, and Euler classes are determined by the formal group law associated to the complex orientation. The question of evenness and regularity of these theories is closely

related to the analysis of [HKR00, Sec. 5]. In particular all examples listed in [HKR00, Sec. 1.3], including Morava E-theories, have even Borel theories. The Borel theory for the Eilenberg-MacLane spectrum $H\mathbb{F}_2$ is not regular, since $e_{\tau^2} = 0 \in H^*(B\mathbb{T}_+, \mathbb{F}_2)$.

3. Evenness and regularity for complex bordism

We denote by **MU** the global complex bordism spectrum as introduced in [Sch18, Ex. 6.1.53], with the canonical complex orientation described there. For every $A \in \mathrm{Ab}$, its associated A-equivariant spectrum is equivalent to tom Dieck's homotopical A-equivariant bordism MU_A. The geometric fixed points $\Phi^A \mathrm{MU}_A$ identify with a sum of even shifts of weak products of the form $(\prod' \mathrm{BU})_+ \otimes \mathrm{MU}$, and are hence free MU-modules concentrated in even degrees with even homotopy groups, see [Sin01, Thm. 4.10] or the appendix for general statements about geometric fixed points of equivariant Thom spectra. By Lemma 2.3, regularity and evenness are hence equivalent, and we show the former.

The main ingredient is the following additional structure on relative localizations, which will be proved in Section 4.

Proposition 3.1. *Let $A \in \mathrm{Ab}$, and $B, C \subseteq A$ closed subgroups such that the map*

$$B/(B \cap C) \to A/C$$

is an isomorphism. Write $\mathcal{V}^B_{B\cap C}$ for the collection of B-characters which are non-trivial on $B \cap C$, and assume a subset $\mathcal{V} \subseteq A^*$ satisfies the following:

(i) The restriction $\mathrm{res}^A_B \colon \mathcal{V} \to \mathcal{V}^B_{B\cap C}$ is bijective.
(ii) Given $V_1, V_2 \in \mathcal{V}$, if $\mathrm{res}^A_{B\cap C} V_1 = \mathrm{res}^A_{B\cap C} V_2$, then $\mathrm{res}^A_C V_1 = \mathrm{res}^A_C V_2$.

Then the restriction map

$$\pi^A_* \mathrm{MU}_A[e_{\mathcal{V}}^{-1}] \to \pi^B_* \mathrm{MU}_B[e_{\mathcal{V}^B_{B\cap C}}^{-1}]$$

has a multiplicative section.

Here, $\pi^A_* \mathrm{MU}_A[e_{\mathcal{V}}^{-1}]$ denotes the ring obtained from $\pi^A_* \mathrm{MU}_A$ by inverting all Euler classes e_V with $V \in \mathcal{V}$, and $\pi^B_* \mathrm{MU}_B[e_{\mathcal{V}^B_{B\cap C}}^{-1}]$ is defined analogously.

Remark 3.2.
- The proof really shows that there exists an E_∞-section to the restriction map on categorical fixed point spectra
$$(\mathrm{MU}_A[e_{\mathcal{V}}^{-1}])^A \to (\mathrm{MU}_B[e_{\mathcal{V}^B_{B\cap C}}^{-1}])^B,$$
whose effect on homotopy groups yields the statement of Proposition 3.1.
- If $B \cap C$ is trivial, and hence $\mathcal{V}^B_{B\cap C}$ and $\mathcal{V}$ are empty, the projection $A \to A/C \cong B$ gives a splitting to the inclusion $B \hookrightarrow A$, and hence the multiplicative section above is a formal consequence of the global functoriality. Proposition 3.1 can hence be seen as a further extension of the global structure of equivariant complex bordism.

Proof of Theorem 2. Let $A \in \mathrm{Ab}$. We show that e_V is regular in $\pi^A_* \mathrm{MU}_A$ – or equivalently that $\mathrm{res}^A_{\ker(V)}$ is surjective – for all characters V of infinite order. Without loss of generality we can assume that $A = A' \times \mathbb{T}$, and V is of the form $W \otimes \tau^n$ for some $W \in (A')^*$ and $n \geq 1$.

We show that every corresponding Euler class $e_{W\otimes\tau^n}$ is regular by induction on n. Set $B = \ker(W \otimes \tau^n)$, $C = 1 \times \mathbb{T}$ and let $\mathcal{V} \subset A^* = (A')^* \times \mathbb{Z}$ denote the collection of characters with second factor of the form τ^i with $1 \leq i < n$, or in other words the collection of characters whose Euler

classes we know by induction are regular. (Note that $\mathcal{V} = \emptyset$ if $n = 1$.) Then $B \cap C = 1 \times C_n$, and one readily checks that Conditions (i) and (ii) of Proposition 3.1 are satisfied.

We want to show that $e_{W\otimes\tau^n}$ is regular. Since we already know that each Euler class for a character in $\mathcal{V}$ is regular, it suffices to show regularity of $e_{W\otimes\tau^n}$ after inverting $e_{\mathcal{V}}$. By Proposition 3.1, the restriction

$$(\pi_*^A \operatorname{MU}_A)[e_{\mathcal{V}}^{-1}] \to (\pi_*^B \operatorname{MU}_B)[e_{\mathcal{V}^B_{B\cap C}}^{-1}]$$

is surjective. The $e_{\mathcal{V}}$-inverted long exact sequence (1) then implies that $e_{W\otimes\tau^n}$ is regular in $(\pi_*^A \operatorname{MU}_A)[e_{\mathcal{V}}^{-1}]$, as desired. This finishes the proof. □

Remark 3.3. Using the spectrum level sections from Remark 3.2 one can use the same argument as above to deduce that the basechange of $\mathbf{MU}$ along any homotopy-commutative MU-algebra R (in other words the Ab-global spectrum $\mathbf{MU} \otimes_{L\,\mathrm{MU}} LR$ with L the inflation functor from non-equivariant spectra to global spectra as in [Sch18, Thm 4.5.1]) is always regular, and even if and only if R is even. For example this shows that each $\mathbf{MU}_{(p)}/(v_0, \dots, v_n)$ is even, and also the basechange to BP.

These generalizations also follow from the theorem shown in [Com96] that each $\pi_*^A \operatorname{MU}_A$ is a free $\pi_* \operatorname{MU}$-module. We do not know how to deduce freeness using the methods of this note.

4. Construction of the section of the Euler-inverted restriction map

In this section we prove Proposition 3.1. Let $A \in \mathrm{Ab}$ with closed subgroups $B, C \subset A$ and collection $\mathcal{V} \subseteq A^*$ satisfying Conditions (i) and (ii) of the proposition. We begin with a few preliminary constructions. Given a character $V \in B^*$ which restricts non-trivially to $B \cap C$, we write $\tilde{V}$ for the unique lift to $\mathcal{V} \subseteq A^*$. We can extend these lifts to all characters of B in the following way: If $B \cap C$ acts trivially on V, we let $\tilde{V}$ be the A-representation given by the pullback of V along the projection

$$A \to A/C \xleftarrow{\cong} B/(B \cap C).$$

We let $\tilde{\mathcal{V}}$ denote the collection of all irreducible A-representations of the form $\tilde{V}$, and note that $\operatorname{res}^A_B \colon \tilde{\mathcal{V}} \to B^*$ is bijective. We further write $\operatorname{Rep}(A)^C_{\tilde{\mathcal{V}}}$ for the $\operatorname{Top}_{A/C}$-enriched category of complex A-representations which decompose into sums of characters in $\tilde{\mathcal{V}}$ and C-equivariant maps between them. We obtain a restriction map

$$\operatorname{Rep}(A)^C_{\tilde{\mathcal{V}}} \to \operatorname{Rep}(B)^{B\cap C} \tag{2}$$

to the $\operatorname{Top}_{B/(B\cap C)}$-enriched category of complex B-representations and $(B \cap C)$-equivariant maps. This map is A/C-equivariant under the identification $B/(B \cap C) \cong A/C$. Then we have:

Lemma 4.1. *The functor* (2) *is an equivalence of* $\operatorname{Top}_{A/C}$*-enriched categories in the sense that it is a homeomorphism on* Hom-A/C*-spaces and the induced functor on the* A/C*-fixed subcategories is essentially surjective.*

Proof. In both A/C-fixed subcategories, every object decomposes as a direct sum of indecomposables, and we saw that the restriction map gives a bijection between those. This implies essential surjectivity. Moreover, given two characters $V_1, V_2 \in \tilde{V}$, we have

$$\operatorname{Hom}_{\operatorname{Rep}(A)^C_{\tilde{\mathcal{V}}}}(V_1, V_2) = \begin{cases} \mathbb{C} & \text{if } \operatorname{res}^A_C V_1 = \operatorname{res}^A_C V_2 \\ 0 & \text{else} \end{cases}$$

and similarly for homomorphisms between $\mathrm{res}^A_B V_1$ and $\mathrm{res}^A_B V_2$ in $\mathrm{Rep}(B)^{B\cap C}$. Hence Condition (ii) implies that res^A_B is an isomorphism on Hom-spaces. □

By choosing an inverse to this equivalence of A/C-fixed subcategories we can now define $\tilde{W}$ for any B-representation W. Then the poset of C-subrepresentations of $\tilde{W}$ bijects to the poset of $(B\cap C)$-subrepresentations of W under the restriction map, and the C-fixed points $\tilde{W}^C$ restrict isomorphically to the $(B\cap C)$-fixed points $W^{B\cap C}$.

We now construct the section explicitly using the orthogonal spectrum model for **MU** introduced in [Sch18, Ex. 6.1.53]. More precisely, we use the unitary analog indexed on complex inner product spaces. In the appendix we sketch a more model-independent version of the construction.

The essential property of **MU** which gives rise to the section is the following observation:

Lemma 4.2. *For all finite dimensional B-representations W, the space level restriction map*

$$\mathrm{MU}_A(\tilde{W})^C \to \mathrm{MU}_B(W)^{B\cap C}$$

is a homeomorphism.

Proof. The based A-space $\mathrm{MU}_A(\tilde{W})$ is the Thom space of the tautological bundle over the Grassmannian of $\dim\tilde{W}$-dimensional complex subspaces of $\tilde{W}^2$. Hence, every non-basepoint corresponds to a pair (V,v) of such a $\dim\tilde{W}$-dimensional subspace V and a vector $v\in V$. Such a point (V,v) is C-fixed if and only if V is a C-subrepresentation and v is a C-fixed point of this subrepresentation. The analogous description holds for $(B\cap C)$-fixed points of $\mathrm{MU}_B(W)^{B\cap C}$. The claim hence follows by Lemma 4.1 and the succeeding paragraph. □

Up to Thom isomorphism, multiplication with each Euler class e_V is given by the map $\mathrm{MU}_B\to S^V\otimes\mathrm{MU}_B$ obtained as the tensor product of the inclusion of $S^0\hookrightarrow S^V$ with MU_B, and similarly for MU_A. It follows that we have isomorphisms

$$\pi_*^B\,\mathrm{MU}_B[e^{-1}_{\mathcal{V}^B_{B\cap C}}]\cong\pi_*^B(S^{\mathcal{U}_B-\mathcal{U}_B^{B\cap C}}\otimes\mathrm{MU}_B)$$

and

$$\pi_*^A\,\mathrm{MU}_A[e_{\mathcal{V}}^{-1}]\cong\pi_*^A(S^{\tilde{\mathcal{U}}_B-\tilde{\mathcal{U}}_B^C}\otimes\mathrm{MU}_A),$$

where $\mathcal{U}_B$ denotes a complete complex B-universe and $\tilde{\mathcal{U}}_B$ its lift to an A-universe constructed above. The based B-space $S^{\mathcal{U}_B-\mathcal{U}_B^{B\cap C}}$ is characterized by the property that the fixed points at a subgroup are contractible if it does not contain $B\cap C$, and equivalent to S^0 otherwise. It follows that for every W the fixed point map yields a bijection

$$[S^{k+W},S^{\mathcal{U}_B-\mathcal{U}_B^{B\cap C}}\otimes\mathrm{MU}_B(W)]^B\xrightarrow{\cong}[S^{k+W^{B\cap C}},\mathrm{MU}_B(W)^{B\cap C}]^{B/(B\cap C)}. \tag{3}$$

Similarly, the fixed points of $S^{\tilde{\mathcal{U}}_B-\tilde{\mathcal{U}}_B^C}$ by every subgroup of A that arises as the isotropy of a point in $S^{k+\tilde{W}}-S^{k+\tilde{W}^C}$ are contractible (since $\oplus_{\mathbb{N}}(\tilde{W}-\tilde{W}^C)$ embeds into $\tilde{\mathcal{U}}_B-\tilde{\mathcal{U}}_B^C$), and hence we obtain a fixed point bijection

$$[S^{k+\tilde{W}},S^{\tilde{\mathcal{U}}_B-\tilde{\mathcal{U}}_B^C}\otimes\mathrm{MU}_A(\tilde{W})]^A\xrightarrow{\cong}[S^{k+\tilde{W}^C},\mathrm{MU}_A(\tilde{W})^C]^{A/C}. \tag{4}$$

Combining the homeomorphism from Lemma 4.2, the homeomorphism $S^{k+\tilde{W}^C}\xrightarrow{\cong}S^{k+W^{B\cap C}}$ and the isomorphism $B/(B\cap C)\cong A/C$, we see that the targets of (3) and (4) are identified under the restriction map. Hence restriction is a bijection

$$[S^{k+\tilde{W}},S^{\tilde{\mathcal{U}}_B-\tilde{\mathcal{U}}_B^C}\otimes\mathrm{MU}_A(\tilde{W})]^A\xrightarrow{\cong}[S^{k+W},S^{\mathcal{U}_B-\mathcal{U}_B^{B\cap C}}\otimes\mathrm{MU}_B(W)]^B,$$

for every finite dimensional B-representation W. Passing to colimits we see that restriction gives an isomorphism

$$\begin{aligned}\operatorname{colim}_{\tilde{W}\subset\tilde{\mathcal{U}}_B}[S^{k+\tilde{W}}, S^{\tilde{\mathcal{U}}_B-\tilde{\mathcal{U}}_B^C}\otimes \mathrm{MU}_A(\tilde{W})]^A &\xrightarrow{\cong} \operatorname{colim}_{W\subset\mathcal{U}_B}[S^{k+W}, S^{\mathcal{U}_B-\mathcal{U}_B^{B\cap C}}\otimes \mathrm{MU}_B(W)]^B\\ &= \pi_k^B(S^{\mathcal{U}_B-\mathcal{U}_B^{B\cap C}}\otimes \mathrm{MU}_B).\end{aligned}$$

The desired section is obtained as the inverse of this isomorphism followed by the map

$$\operatorname{colim}_{\tilde{W}\subset\tilde{\mathcal{U}}_B}[S^{k+\tilde{W}}, S^{\tilde{\mathcal{U}}_B-\tilde{\mathcal{U}}_B^C}\otimes \mathrm{MU}_A(\tilde{W})]^A \to \pi_k^A(S^{\tilde{\mathcal{U}}_B-\tilde{\mathcal{U}}_B^C}\otimes \mathrm{MU}_A)$$

induced by an inclusion of A-universes $\tilde{\mathcal{U}}_B \hookrightarrow \mathcal{U}_A$. This finishes the proof.

Remark 4.3. We note that we showed the following more general statement: Let E be a unitary spectrum such that the restriction map

$$E_A(\tilde{W})^C \to E_B(W)^{B\cap C}$$

is an equivalence for all B-representations W. Then the above construction yields a section to the restriction map

$$\pi_*^A(S^{\tilde{\mathcal{U}}_B-\tilde{\mathcal{U}}_B^C}\otimes E_A) \to \pi_*^B(S^{\mathcal{U}_B-\mathcal{U}_B^{B\cap C}}\otimes E_B),$$

which is multiplicative if E is a ring spectrum. More generally the statement is true in representation gradings of the form $\tilde{W}_1-\tilde{W}_2$. Other examples of such E are the geometric global complex bordism spectrum $\mathbf{mU}$, the unitary analog of $\mathbf{mO}$ defined in [Sch18, Ex. 6.1.24], and the global sphere spectrum $\mathbb{S}$.

Appendix: Categorical proof of the existence of sections via Thom spectra in incomplete universes

In this appendix we describe a sketch of a more categorical and model-independent version of the argument from Section 4. We hope that the reader finds this useful, even though not all properties of equivariant Thom spectra for incomplete universes that are used below are recorded in the literature. For an ∞-categorical treatment of equivariant Thom spectra indexed on complete universes, we refer to [BL26]. It will be more convenient to work with the periodic version $\mathrm{MUP}_A = \bigvee_{i\in\mathbb{Z}} S^{2i}\otimes \mathrm{MU}_A$ (see [Sch18, Ex. 6.1.53]) of complex bordism. Restriction to the degree 0 component gives the analog for MU_A. Given an abelian compact Lie group A and a complex A-universe $\mathcal{U}$, we write $\mathrm{Sp}_{A,\mathcal{U}}$ for the ∞-category of A-spectra indexed on $\mathcal{U}$ with suspension spectrum functor

$$\Sigma^\infty\colon \mathcal{S}_{A,*}\to \mathrm{Sp}_{A,\mathcal{U}},$$

which is initial among functors in $\mathrm{Pr}_L^\otimes$ sending all representation spheres S^W for finite-dimensional $W\subseteq\mathcal{U}$ to invertible objects. When $\mathcal{U}$ is a complete A-universe, we simply write Sp_A. We obtain a symmetric monoidal change of universe functor

$$L_{\mathcal{U}}\colon \mathrm{Sp}_{A,\mathcal{U}}\to \mathrm{Sp}_A\,.$$

Let C be a closed subgroup of A. We write

$$(-)^{C\subseteq A,\mathcal{U}}\colon \mathrm{Sp}_{A,\mathcal{U}}\to \mathrm{Sp}_{A/C,\mathcal{U}^C}$$

for the lax monoidal relative categorical fixed point functor, i.e., the right adjoint of the inflation functor

$$q^*\colon \mathrm{Sp}_{A/C,\mathcal{U}^C}\to \mathrm{Sp}_{A,\mathcal{U}^C}\to \mathrm{Sp}_{A,\mathcal{U}}\,.$$

We write $\mathrm{BUP}_A\simeq\Omega^\infty\mathrm{KU}_A$ for the classifying space of virtual complex A-bundles. More generally, we obtain a variant $\mathrm{BUP}_{A,\mathcal{U}}$ by restricting to complex vector bundles for which all fibers

over a point of isotropy K embed in $\mathrm{res}^A_K \mathcal{U}$. Explicitly, one model for $\mathrm{BUP}_{A,\mathcal{U}}$ is given by the group completion of the E_∞-A-monoid $\bigsqcup_{n\in\mathbb{N}} \mathrm{Gr}_n(\mathcal{U})$ of Grassmannians of n-dimensional subspaces of $\mathcal{U}$. We obtain a Thom spectrum functor

$$\mathrm{Th}_{\mathcal{U}} \colon \mathcal{S}_A/\,\mathrm{BUP}_{A,\mathcal{U}} \to \mathrm{Sp}_{A,\mathcal{U}}$$

as the unique parametrized colimit preserving extension of the J-homomorphism

$$\mathrm{BUP}_{A,\mathcal{U}} \to \mathrm{pic}(\mathrm{Sp}_{A,\mathcal{U}}) \hookrightarrow \mathrm{Sp}_{A,\mathcal{U}},$$

see [BL26, Part III] for the more general orthogonal version (in the case where $\mathcal{U}$ is complete). We note that the restriction on the fibers guarantees that the restriction of the J-homomorphism $\mathrm{BUP}_A \to \mathrm{pic}(\mathrm{Sp}_A)$ along $\mathrm{BUP}_{A,\mathcal{U}} \to \mathrm{BUP}_A$ indeed factors through $\mathrm{pic}(\mathrm{Sp}_{A,\mathcal{U}})$. Since L_U is a left adjoint, it preserves parametrized colimits, and hence we have

$$L_{\mathcal{U}}(\mathrm{Th}_{\mathcal{U}}(X \to \mathrm{BUP}_{A,\mathcal{U}})) \simeq \mathrm{Th}(X \to \mathrm{BUP}_{A,\mathcal{U}} \to \mathrm{BUP}_A).$$

We set

$$\mathrm{MUP}_{A,\mathcal{U}} = \mathrm{Th}_{\mathcal{U}}(\mathrm{id}\colon \mathrm{BUP}_{A,\mathcal{U}} \to \mathrm{BUP}_{A,\mathcal{U}}) \in \mathrm{Sp}_{A,\mathcal{U}},$$

which comes with a natural comparison map

$$L_{\mathcal{U}}\,\mathrm{MUP}_{A,\mathcal{U}} \simeq \mathrm{Th}(\mathrm{BUP}_{A,\mathcal{U}} \to \mathrm{BUP}_A) \to \mathrm{Th}(\mathrm{BUP}_A) \simeq \mathrm{MUP}_A\,.$$

We now return to the setting of Section 4 and use the same notation of A, B, C and $\mathcal{V}$. In particular, $\tilde{\mathcal{U}}_B$ denotes the extension of $\mathcal{U}_B$ to an A-universe in the sense constructed there, so that the C-fixed points $\tilde{\mathcal{U}}_B^C$ agree with the $B\cap C$-fixed points $\mathcal{U}_B^{B\cap C}$. Then we have:

Lemma 4.4. *The restriction map*

$$(S^{\tilde{\mathcal{U}}_B - \tilde{\mathcal{U}}_B^C} \otimes \mathrm{MUP}_{A,\tilde{U}_B})^{C\subseteq A} \to (S^{\mathcal{U}_B - \mathcal{U}_B^{B\cap C}} \otimes \mathrm{MUP}_{B,\mathcal{U}_B})^{B\cap C\subseteq B}$$

is an equivalence of $A/C \cong B/(B\cap C)$-spectra.

Proof. By the universal property, the ∞-categories $\mathrm{Sp}_{A,\mathcal{U}}$ come with symmetric monoidal geometric fixed point functors

$$\Phi_{\mathcal{U}}^{C\subseteq A}\colon \mathrm{Sp}_{A,\mathcal{U}} \to \mathrm{Sp}_{A/C,\mathcal{U}^C}$$

such that $\Phi_{\mathcal{U}}^{C\subseteq A}\Sigma^\infty A \simeq \Sigma^\infty A^C$ and $L_{\mathcal{U}^C} \circ \Phi_{\mathcal{U}}^{C\subseteq A} \simeq \Phi^{C\subseteq A} \circ L_{\mathcal{U}}$. There is a natural comparison map

$$\phi^C \colon X^{C\subseteq A,\mathcal{U}} \to \Phi^{C\subseteq A,\mathcal{U}} X$$

relating categorical and geometric fixed points.

Lemma 4.5. *For any complex A-universe $\mathcal{U}$ the map*

$$\phi^C \colon (S^{\mathcal{U}-\mathcal{U}^C} \otimes X)^{C\subseteq A,\mathcal{U}} \to \Phi^{C\subseteq A,\mathcal{U}}(S^{\mathcal{U}-\mathcal{U}^C} \otimes X) \simeq \Phi^{C\subseteq A,\mathcal{U}} X$$

is an equivalence of $\mathrm{Sp}_{A/C,\mathcal{U}^C}$-spectra.

Proof. This is the incomplete universe analog of [MM02, Prop. V.4.17]. □

Applied to our situation it follows that the maps

$$\phi^C \colon (S^{\tilde{\mathcal{U}}_B - \tilde{\mathcal{U}}_B^C} \otimes \mathrm{MUP}_{A,\tilde{\mathcal{U}}_B})^{C\subseteq A} \to \Phi^{C\subseteq A,\tilde{\mathcal{U}}_B}\,\mathrm{MUP}_{A,\tilde{\mathcal{U}}_B}$$

and

$$\phi^{B\cap C} \colon (S^{\mathcal{U}_B - \mathcal{U}_B^{B\cap C}} \otimes \mathrm{MUP}_{B,\mathcal{U}_B})^{B\cap C\subseteq B} \to \Phi^{(B\cap C)\subseteq B}(\mathrm{MUP}_{B,\mathcal{U}_B})$$

are equivalences, and hence the restriction map of Lemma 4.4 is of the form

$$\Phi^{C\subseteq A,\tilde{\mathcal{U}}_B}\,\mathrm{MUP}_{A,\tilde{\mathcal{U}}_B} \to \Phi^{(B\cap C)\subseteq B}(\mathrm{MUP}_{B,\mathcal{U}_B}). \tag{5}$$

Moreover, geometric fixed points of Thom spectra are given by the formula

$$\Phi^{C\subseteq A}(\mathrm{Th}_{\mathcal{U}}(X\to \mathrm{BUP}_{A,\mathcal{U}})) \simeq \mathrm{Th}_{\mathcal{U}^C}(X^{C\subseteq A}\to \mathrm{BUP}_{A,\widetilde{\mathcal{U}}}^{C\subseteq A}\to \mathrm{BUP}_{A/C,\mathcal{U}^C}).$$

Here, the map $\mathrm{BUP}_{A,\widetilde{\mathcal{U}}}^{C\subseteq A}\to \mathrm{BUP}_{A/C,\mathcal{U}^C}$ classifies taking fibrewise C-fixed points of a virtual A-equivariant complex vector bundle over an A-space on which C acts trivially. Unravelling the definitions, we hence find that the restriction map (5) is equivalent to the Thomification of the restriction map

$$\mathrm{BUP}_{A,\tilde{\mathcal{U}}_B}^{C\subseteq A}\to \mathrm{BUP}_{B,\mathcal{U}_B}^{B\cap C\subseteq B},$$

where we have again identified $B/(B\cap C)\cong A/C$. We claim that this restriction map is an equivalence. Indeed, this follows essentially by Lemma 4.1, since $\mathrm{BUP}_{A,\tilde{\mathcal{U}}_B}^{C\subseteq A}$ is equivalent to the group completion of the A/C-E_∞-monoid of finite dimensional complex C-subrepresentations of $\tilde{\mathcal{U}}_B$, and similarly for $\mathrm{BUP}_{B,\mathcal{U}_B}^{B\cap C\subseteq B}$. Said differently, we have equivalences

$$\mathrm{BUP}_{A,\tilde{\mathcal{U}}_B}^{C\subseteq A}\simeq \prod_{\mathrm{res}_C^A\,\mathcal{V}\cup\{\epsilon\}}{}' \mathrm{BUP}_{A/C}$$

and

$$\mathrm{BUP}_{B,\mathcal{U}_B}^{B\cap C\subseteq B}\simeq \prod_{(B\cap C)^*}{}' \mathrm{BUP}_{B/(B\cap C)}\,.$$

So the claim follows from the fact that the restriction map $\mathrm{res}^C_{B\cap C}\colon \mathrm{res}^A_C\,\mathcal{V}\cup\{\epsilon\}\to (B\cap C)^*$ is a bijection by Conditions (i) and (ii), which finishes the proof of the lemma. □

We obtain a commutative diagram of E_∞-ring spectra:

$$\begin{array}{ccccc}
(S^{\tilde{\mathcal{U}}_B-\tilde{\mathcal{U}}_B^C}\otimes \mathrm{MUP}_{A,\tilde{\mathcal{U}}_B})^{C\subseteq A} & \xrightarrow{L_{\tilde{\mathcal{U}}_B}} & (L_{\tilde{\mathcal{U}}_B}(S^{\tilde{\mathcal{U}}_B-\tilde{\mathcal{U}}_B^C}\otimes \mathrm{MUP}_{A,\tilde{\mathcal{U}}_B}))^{C\subseteq A} & \longrightarrow & (S^{\tilde{\mathcal{U}}_B-\tilde{\mathcal{U}}_B^C}\otimes \mathrm{MUP}_A)^{C\subseteq A} \\
& \searrow^{\simeq} & \downarrow{\scriptstyle \mathrm{res}^A_B} & \swarrow_{\mathrm{res}^A_B} & \\
& & (S^{\mathcal{U}_B-\mathcal{U}_B^{B\cap C}}\otimes \mathrm{MUP}_{B,\mathcal{U}_B})^{B\cap C\subseteq B} & &
\end{array}$$

The lower line consists only of one object since $\tilde{\mathcal{U}}_B$ already restricts to a complete B-universe and hence the change of universe at A does not have an effect on underlying B-spectra. The left hand vertical map is an equivalence by Lemma 4.4. Passing to $B/(B\cap C)\cong A/C$-equivariant homotopy groups yields the multiplicative section to the right hand vertical restriction map and proves Proposition 3.1.

References

[AP25] K. Allen and L. Piessevaux. Synthetic equivariant spectra for finite abelian groups and motivic homotopy theory, 2025. URL: https://arxiv.org/abs/2510.20197.

[BL26] E. Brink and T. Lenz. Global ∞-categories and global Thom spectra, 2026. URL: https://arxiv.org/abs/2608.28504.

[Com96] G. Comezaña. *Calculations in complex equivariant bordism.* in: J.P.May(ed.) Equivariant homotopy and cohomology theory, CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996. doi:10.1090/cbms/091.

[Hah26] J. Hahn. Algebraic cohomology of ring spectra. In *Proceedings of the International Congress of Mathematicians 2026 – Volume 4: Invited Lectures (Sections 5–8)*, pages 333–342, 2026. doi:10.1137/25M1807290.

[Hau22] M. Hausmann. Global group laws and equivariant bordism rings. *Ann. of Math. (2)*, 195(3):841–910, 2022. doi:10.4007/annals.2022.195.3.2.

[HKR00] M. Hopkins, N. Kuhn, and D. Ravenel. Generalized group characters and complex oriented cohomology theories. *J. Amer. Math. Soc.*, 13(3):553–594, 2000. doi:10.1090/S0894-0347-00-00332-5.

[HRW26] J. Hahn, A. Raksit, and D. Wilson. A motivic filtration on the topological cyclic homology of commutative ring spectra. *arXiv preprint, to appear in Annals of Mathematics*, 2026. URL: https://arxiv.org/abs/2206.11208.

[LNP25] S. Linskens, D. Nardin, and L. Pol. Global homotopy theory via partially lax limits. *Geom. Topol.*, 29(3):1345–1440, 2025. doi:10.2140/gt.2025.29.1345.

[Löf73] P. Löffler. Equivariant unitary cobordism and classifying spaces. *Proceedings of the International Symposium on Topology and its Applications (Budva, 1972)*, pages 158–160, 1973.

[Löf74] P. Löffler. Bordismengruppen unitärer Torusmannigfaltigkeiten. *Manuscripta Math.*, 12:307–327, 1974. doi:10.1007/BF01171078.

[Mil60] J. W. Milnor. On the cobordism ring Ω^* and a complex analogue. I. *Amer. J. Math.*, 82:505–521, 1960. doi:10.2307/2372970.

[MM02] M. A. Mandell and J. P. May. Equivariant orthogonal spectra and S-modules. *Mem. Amer. Math. Soc.*, 159(755):x+108, 2002. doi:10.1090/memo/0755.

[Sch18] S. Schwede. *Global homotopy theory*, volume 34 of *New Math. Monogr.* Cambridge University Press, Cambridge, 2018. doi:10.1017/9781108349161.

[Sin01] D. Sinha. Computations of complex equivariant bordism rings. *Amer. J. Math.*, 123(4):577–605, 2001. URL: http://muse.jhu.edu/journals/american_journal_of_mathematics/v123/123.4sinha.pdf.

[tD70] T. tom Dieck. Bordism of G-manifolds and integrality theorems. *Topology*, 9:345–358, 1970. doi:10.1016/0040-9383(70)90058-3.