Some Consistency Strength Analyses using Higher Core Models

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Chapter 1

Introduction

'Twas brillig, and the slythy toves
did gyre and gimble in the wabe.
All mimsy were the borogoves
and the mome rath outgrabe.

(Lewis Carroll, Jabberwocky)

Set Theory, besides having its origins[1] in Cantor's investigations into the convergence of trigonometric series, has been closely linked with attempts to formulate a viable Foundation of Mathematics. At the turn of the century Zermelo, with the purpose of clearly exhibiting the assumptions used in his proof of the Well-Ordering Theorem, provided the basis for what has now become the standard axiomatisation of Set Theory, namely ZF. From the beginning, the question of the consistency of this (or any) system of axioms loomed large. Hilbert [Hil00] had included the consistency of Arithmetic in his famous list of open problems at the international congress of mathematicians in Paris in 1900. The rest is history, one is tempted to say: Gödel, in his 1931 paper [Göd31], showed that Hilbert's goal, i. e., proving the consistency of ZF by purely finitary means, could not be achieved. The consistency of any system strong enough to code Peano arithmetic cannot be proved even by means of the system itself, let alone by finitary reasoning, as

[1] For a detailed survey of the early history of Set Theory and its ramifications — which we have not attempted here — cf. [Kan96].

demanded by Hilbert. In fact, for any such system, there will be undecidable sentences. The prime example is, of course, the Continuum Hypothesis, which dates back to Cantor's original investigations of the universe of sets. Gödel [Göd38] gave one half of the independence proof, proving the consistency of $\mathrm{ZF}+\mathrm{CH}$ from Con(ZF), and Cohen [Coh63], [Coh64] showed that the negation of CH is equally relatively consistent.

These two proofs already contain the two main methods for arriving at relative consistency results: Gödel constructed an *inner model*, i. e., a class $M \subseteq V$ which satisfies all the axioms of ZF, together with CH. His model, L, can be seen as the prototype of the whole family of *core models* that were to be developed later on. Cohen, on the other hand, by his method of *forcing*, constructed an extension of the universe $\mathrm{V}[G]$, in which the ZF-axioms hold, as well as the negation of CH.

All consistency proofs must be *relative*, i. e., they have to assume the consistency of some set of axioms at the outset, possibly ZF, possibly $\mathrm{ZF}+\Gamma$, where Γ is some (set of) sentence(s). Then they can construct a new model in which the statement in question, say CH or Φ, holds. One class of extensions of ZF, that of *large cardinal axioms*, has proved to be particularly fruitful for these investigations.[2] They are perhaps best characterized by positing the existence of some ordinal with special properties or that of some elementary map from the universe of sets to some structure M. What makes them attractive is the fact that they form a nearly linear scale against which one can gauge the *consistency strength* of various other, say, combinatorial statements. Thus, taking ZFC as the base system, if Λ_1 and Λ_2 are large cardinal axioms and Φ is some combinatorial statement, then Λ_1 is an upper bound to the consistency strength of Φ if one can show that $\mathrm{Con}(\mathrm{ZFC}+\Lambda_1)$ implies $\mathrm{Con}(\mathrm{ZFC}+\Phi)$. Analogously, Λ_2 will be a lower bound to the consistency strength of Φ if one can construct, starting from a model of $\mathrm{ZFC}+\Phi$, a model of $\mathrm{ZFC}+\Lambda_2$. As exemplified by Gödel's and Cohen's results on CH (although their proofs did not require any axioms beyond ZF), upper bounds are usually constructed by the method of forcing, where the forcing exploits the properties of the large cardinals, while the lower bounds make use of

[2] [Kan94] gives a comprehensive introduction to large cardinals including numerous historical remarks.

inner models.

Core models, a special type of inner models, where invented by Jensen and Dodd [DJ81]. They go back to Jensen's investigation of Gödel's model L of constructible sets. Gödel's fundamental idea was to construct the model layer by layer, adding sets which are definable from parameters previously constructed at each stage. Jensen [Jen72] took a closer look at this layering or definability and thus arrived at more precise statements about the (fine) structure of the resulting model. Using this finestructure, he succeeded together with Dodd in constructing K^{DJ}, the first core model. It lies somewhere between L and $\mathrm{L}[U]$, the canonical inner model for a measurable cardinal.

The building blocks of this model are initial segments of the final model called mice. They play a central rôle in all of core model theory. Subsequently, extensions of this concept of mouse led to new core models encompassing ever larger large cardinal hypotheses.[3] Some common properties are their construction from the bottom up along the ordinals, their rigidity (the existence of an elementary embedding from K to K being the least large cardinal axiom inconsistent with K) and some sort of covering lemma, which asserts that — assuming the absence of the corresponding large cardinal in V — the model K is close to V. Mitchell [Mit84] described the core model for sequences of measures, Jensen [Jen8x] among others that for measures of order zero. Schindler [Sch96], building on work of Jensen and Koepke [Koe89], constructed a core model up to a strong cardinal. Transcending the boundary of linear iterations using iteration trees, Steel [MS94], [Ste96] extended the theory so as to accomodate even larger hypotheses, currently somewhere in the region of limits of Woodin cardinals.

A common approach to consistency strength analysis using core models is the following. Suppose we have some combinatorial statement Φ for which we want to give a lower bound of its consistency strength, say Λ, where Λ is some large cardinal axiom. Assume that Λ does *not* hold in V. Then a covering lemma will hold for a suitable core model K. One then tries to show that the close connection between the core model K and the surrounding universe V contradicts Φ, in the sense that Φ disrupts the "constructible" nature of V imposed by K.

[3]For an introduction and an approximate definition of this class of models cf. [Mit94].

In Chapter 2 we review the theory of the so-called *short core models*. This theory has been developed in detail in [Koe83], and [Koe88] contains an introduction to the main concepts and basic results, which we summarise without proof. Short core models allow inner models for sequences of measurable cardinals, provided that the length of the sequence is less than the least measurable. This assumption greatly simplifies the theory of iterations, as the number of measures in a mouse does not change (increase, that is) when one takes an ultrapower.

Chapter 3 takes another look at Chang's Conjecture, CC, and shows that it is equivalent to an apparently stronger version, which we call Chang's Conjecture at club-many points, $\mathrm{CC}^{\mathrm{club}}$. Silver [Sil71] had shown that CC can be forced to hold in a generic extension of the universe, using a variant of the Levy collapse, now known as the Silver collapse. Later, Donder and Levinski [DL89] and Baumgartner [Bau91] showed that the same result can be achieved using the original Levy collapse. We modify this approach to show that in the same generic extension also $\mathrm{CC}^{\mathrm{club}}$ holds.

In Chapter 4 a combinatorial principle closely related to Chang's Conjecture is considered, the so-called Transversal Hypothesis, TH. Usually, the interest lies in the negation of this hypothesis, as this is a consequence of CC. The exact consistency strength of $\neg\,\mathrm{TH}$ at ω_1 has been determined by Donder and Levinski [DL89] in terms of certain game principles. However, at higher cardinals the situation is less clear. In fact, there is a large gap between the lower and upper bounds for the consistency of CC at higher cardinals. So far, a huge cardinal (cf. [For82]) has been needed to force, say, $\langle\aleph_4,\aleph_3\rangle \twoheadrightarrow \langle\aleph_3,\aleph_2\rangle$, whereas Schindler [Sch96] gets a strong cardinal as a lower bound for the consistency strength of this statement (together with $2^{\aleph_1}=\aleph_2$). We in turn consider a variant of TH, $\mathrm{TH}^{\mathrm{stat}}$, concentrating on a stationary set, and give a lower bound for the consistency strength of $\neg\,\mathrm{TH}^{\mathrm{stat}}$ at cardinals beyond $\aleph_1$, using the short core models of Chapter 2. Just as $\neg\,\mathrm{TH}$ is implied by CC, $\neg\,\mathrm{TH}^{\mathrm{stat}}$ is a consequence of $\mathrm{CC}^{\mathrm{club}}$, considered in Chapter 3. That chapter thus serves to make the consistency (relative to some large cardinal, of course) of $\mathrm{TH}^{\mathrm{stat}}$ plausible.

Chapter 5 lays the ground for another consistency strength result. To get higher lower bounds than just 0^{long} (the "sharp" for short core models), we need to consider higher core models. For our purposes, core models up to a strong cardinal will do nicely. The finestructure for these models has been developed in [Koe89], and the actual core model construction is carried through in [Sch96]. As in Chapter 2, we present a brief summary of the main concepts and results.

In Chapter 6 we calculate an estimate for the length of certain short iterations of mice. This bound will play a crucial rôle in the following chapter. We are only able to give this bound under a more restrictive assumption than just the absence of an inner model for a strong cardinal, which limits the possible application in the subsequent consistency strength analysis. However, even though our assumption is perhaps more restrictive than ultimately necessary, some bound strictly below a strong cardinal is necessary, as a short "counter-example" at the end of the chapter shows.

Finally, in Chapter 7 we compute a new lower bound for the consistency strength of the existence of irregular ultrafilters. Regular ultrafilters were first considered in Model Theory. Keisler showed that they yield ultrapowers of maximal cardinality. However, not all ultrafilters need to be regular. A measure on a measurable cardinal, for example, will always be fully irregular. More interesting, though, is the question whether irregular ultrafilters can exist on "small" or successor cardinals, $\aleph_1$ or $\aleph_2$, say. Prikry [Pri70] showed that in L, all ultrafilters on ω_1 (in fact, on any successor cardinal) are regular. Ketonen [Ket76] then showed that if there is, on some regular cardinal, a weakly normal ultrafilter which is fully irregular, then $0^{\#}$ exists. This result was later improved by Jensen [DJK81], who proved that the existence of an inner model for a measurable cardinal can be deduced. Kanamori [Kan76] showed that if the ultrafilter lives on a successor cardinal, then the weak normality requirement can be dropped. Donder [Don88] investigated the matter at singular cardinals: if there is a uniform, non-regular ultrafilter on κ and κ is singular, then there is an inner model for a measurable cardinal. Also, if κ is regular and $(\kappa^+)^{\mathrm{K^{DJ}}} = \kappa^+$, then the same conclusion holds. Thus in the Dodd-Jensen core model $\mathrm{K^{DJ}}$, all uniform ultrafilters are regular.

Not too much is known about the upper bound for the consistency strength of (fully) irregular ultrafilters over small cardinals. Magidor [Mag79] used a forcing starting from a huge cardinal to get a model in which ω_3 carries an ultrafilter that is not (ω_1, ω_3)-regular. Also, he constructed a model in which ω_2 carries an ultrafilter that is not (ω, ω_2)-regular. This not quite full irregularity, though. Laver [Lav82] showed that in a model constructed by Woodin, ω_1 carries an irregular ultrafilter. The hypothesis used for this construction was "$\mathrm{ZF} + \mathrm{AD}_{\mathbb{R}} + \Theta$ regular"[4]. Finally, Foreman, Magidor, and Shelah [FMS88] got an upper bound from large cardinals only: Given an infinite cardinal μ and $\kappa > \mu$ huge, it is consistent that μ^+ carries a fully irregular ultrafilter.[5]

We are able to improve the lower bound of the Jensen result [DJK81] to "There exists an inner model with $o^M(\mu) \geqslant \omega$". Were it not for the restriction of the estimate of the length of short iterations from Chapter 6, our proof would be able to cope with any hypothesis up to a strong cardinal. It is for this reason that we have written up the proof in such a way that it would go through for any such large hypothesis without change, provided this estimate were improved. It is in this vein that we have also included a presentation of the so-called Gitik Game, which permits the reconstruction of certain extenders used in "long" iterations (cf. [Git93]). Given our assumptions it would not have been truly necessary. However, we consider it an interesting method in its own right, apart from its value in pinpointing the exact location of the limitation of our proof. One should also note that our current assumptions would permit further simplifications to the proof, e. g., there is obviously no need to consider overlapped cardinals at all.

A word on notation: our notation adheres to the set theoretic standard, such as that employed in [Kan94]. Thus H_μ denotes the collection of the sets of hereditary cardinality less than μ, where the hereditary cardinality of a set is the cardinality of its transitive closure. The cofinality of an ordinal α

[4] Here, $\Theta = \sup\{\xi \mid \exists f : {}^\omega\omega \to \xi \text{ onto}\}$. The theory "$\mathrm{AD}_{\mathbb{R}} + \Theta$ regular" is the strongest determinacy type hypothesis known, much stronger than $\mathrm{AD}_{\mathbb{R}}$ alone (whose consistency strength is within the limits of current inner model theory). Woodin has shown that the axiom I1 is an upper bound for the strength of "$\mathrm{AD}_{\mathbb{R}} + \Theta$ regular", where I1 is the statement that there is an elementary embedding $j : V_{\delta+1} \to V_{\delta+1}$.

[5] Huberich [Hub94] extended this result to show that a measurable cardinal alone is sufficient to force the existence of a fully irregular ultrafilter on a non-measurable weakly compact cardinal. This puts a bound on the possibility of irregular ultrafilters on limit cardinals having high consistency strengths.

is denoted by $\mathrm{cf}(\alpha)$, the ordertype of a set x by $\mathrm{otp}(x)$, its cardinality by $\mathrm{card}(x)$, or, in some cases, by $\overline{\overline{x}}$. The least upper bound of a set of ordinals is $\mathrm{lub}(x)$. On denotes the class of all ordinals, Lim that of all limit ordinals, Card that of all cardinals, Reg that of all regular cardinals.

A set $x \subseteq \kappa$ is called γ-club if it is unbounded in κ and closed under limits of cofinality at least γ. The set of all subsets of κ of size λ is denoted by $[\kappa]^{\lambda}$, that of all subsets of κ of size less than λ by $[\kappa]^{<\lambda}$, accordingly.

If f is a function (a fact denoted by $\mathrm{fun}(f)$), then $\mathrm{dom}(f)$ stands for its domain and $\mathrm{rge}(f)$ its range. $f''x = \{f(y) \mid y \in x\}$ is the image of x under f and $f \restriction x = \{\langle y, f(y)\rangle \mid y \in x\}$ is the restriction of f to x. If π is an embedding from M to N, then $\mathrm{crit}(\pi)$ denotes the critical point of π, that is the least ordinal moved by π.

$U \subseteq \mathcal{P}(\kappa)$ is an ultrafilter if

- i) it is closed under finite intersections and supersets, and if
- ii) it contains κ, and if
- iii) for any $x \subseteq \kappa$ it contains either x or $\kappa \setminus x$.

U is γ-complete, for some $\gamma \leqslant \kappa$, if U contains the intersection of any β-many elements of U, for any $\beta < \gamma$.

Chapter 2

Short Core Models

Alice thought this must be the right way of speaking to a mouse: she had never done such a thing before, but she remembered having seen in her brother's Latin Grammar, "A mouse – of a mouse – to a mouse – a mouse – O mouse!"

(Lewis Carroll, Alice's Adventures in Wonderland, Chapter II)

The theory of short core models has been developed in [Koe83]. The paper [Koe88] contains a concise introduction to the main concepts and basic results, of which we cite a number without proof for future reference. The numbering from [Koe88] is given in parentheses.

$\mathrm{J}_\alpha[A]$ denotes the α-th level of the relativized Jensen-hierarchy, and $<_{\mathrm{J}_\alpha[A]}$ its canonical well-ordering. A class D is *simple* iff every element x of D is of the form $\langle\kappa, a\rangle$, where κ is an ordinal and $a \subseteq \kappa$, and $\langle\kappa, a\rangle \in D$ implies $\langle\kappa, \kappa\rangle \in D$. For simple D define $\mathrm{dom}(D) := \{\kappa \mid \langle\kappa, \kappa\rangle \in D\}$, $D(\kappa) := \{a \mid \langle\kappa, a\rangle \in D\}$, and $D \upharpoonright X := \{\langle\kappa, a\rangle \mid \kappa \in X\}$.

U is a *measure* on κ iff U is a non-principal, κ-complete, normal ultrafilter on κ. $\mathcal{U}$ is a *sequence of measures* iff $\mathcal{U}$ is simple and $\mathcal{U}(\kappa)$ is a measure on κ for every $\kappa \in \mathrm{dom}(\mathcal{U})$.

2.1 Definition (2.1) Let D be a simple predicate. $M = \mathrm{J}_\alpha[\mathcal{U}, D]$ is a *premouse over* D iff $\mathcal{U}$ is simple, $\sup\mathrm{dom}(D) < \min\mathrm{dom}(\mathcal{U})$, and $M \models \mathcal{U}$ is a sequence of measures. Then $\mathrm{meas}(M) := \mathrm{dom}(\mathcal{U}) \cap M$ is the *set of measurables* in M, and $\mathrm{lp}(M) := \mathrm{H}_\kappa$, where $\kappa = \min\mathrm{meas}(M)$, is the *low part* of M (setting $\mathrm{lp}(M) := M$ if $\mathrm{meas}(M) = \emptyset$).

We omit the detailed definitions of *ultrapowers*, *iterations*, and *iterability*, as well as the relevant criteria for iterability ([Koe88, Definitions and Lemmas 2.2 – 2.8]).

2.2 Definition (2.9) Let $M = \mathrm{J}_\alpha[\mathcal{U}, D]$ be a premouse over D. M is called *short* if *i)* either $D = \emptyset$ and $\mathrm{otp}\,\mathrm{meas}(M \cap \gamma) < \min\mathrm{meas}(M)$ for all $\gamma < \omega\alpha$ or *ii)* $D \neq \emptyset$ and $\mathrm{otp}\,\mathrm{meas}(M) \leqslant \min\mathrm{dom}(D)$.

A *D-premouse* is a short premouse over D, and a *D-mouse* is an iterable short premouse over D.

The main advantage of dealing with short premice is that the number of measures present in a premouse does not change when one takes an ultrapower: $\mathrm{meas}(\mathrm{Ult}(M, U)) = \pi''(\mathrm{meas}(M))$.

2.3 Definition (2.13) "0^{long} exists" means that there is an iterable premouse over $\emptyset$ which is not short. $\neg 0^{\mathrm{long}}$ is taken to abbreviate the statement "0^{long} does not exist".

2.4 Definition (3.1) Let D be simple such that $D = \emptyset$ or $\mathrm{otp}\,\mathrm{dom}(D) \leqslant \min\mathrm{dom}(D)$. Define the class $K[D]$ as

$$K[D] := \bigcup\{\mathrm{lp}(M) \mid M \text{ is a } D\text{-mouse}\}.$$

For $\alpha \in \mathrm{On}$ set $K_\alpha[D] := \mathrm{H}_\alpha^{K[D]}$.

2.5 Theorem (3.2) *$K[D]$ is a transitive inner model of ZFC. $\bar{D} := D \cap K[D] \in K[D]$ and $K[D] \models V = K[\bar{D}]$. If $\alpha > \sup\mathrm{dom}(D)$ is an uncountable cardinal in $K[D]$, then $K_\alpha[D] \models V = K[\bar{D}]$.*

2.6 Lemma (3.3) *Let Q be a transitive model of a sufficiently large finite part of* $\mathrm{ZFC} + \mathrm{V} = K[\bar{D}]$, *where $\bar{D} = D \cap Q \in Q$. Assume* $\mathrm{dom}(\bar{D}) =$

$\mathrm{dom}(D)$ *and that* $\bar{D}$*-mice are absolute for* Q*. Then* $Q \subseteq K_\alpha[D]$*, where* $\alpha = \mathrm{On} \cap Q \leqslant \infty$*.*

2.7 Theorem (3.4) *For* $x,\ y \in K[D]$ *set* $x \leqslant_D y$ *iff* $x \leqslant_M y$ *for every* D*-mouse* M *such that* $x, y \in \mathrm{lp}(M)$*. Then* $\leqslant_D$ *is a well-ordering of* $K[D]$*.*

2.8 Definition (3.6) A model $K[\mathcal{U}]$ is called a *short core model* if $K[\mathcal{U}] \vDash \mathcal{U}$ is a sequence of measures. A set $\mathcal{U}$ such that $K[\mathcal{U}]$ is a short core model is called *strong*.

2.9 Definition (3.8) Let $\mathcal{U}$, $\mathcal{U}'$ be strong. Set $\mathcal{U} \leqslant_e \mathcal{U}'$ if $\mathcal{U} = \mathcal{U}' \restriction \eta$ for some $\eta \in \mathrm{On}$, i. e., if $\mathcal{U}'$ is an *end-extension* of $\mathcal{U}$. Set $\mathcal{U} <_e \mathcal{U}'$ if $\mathcal{U} \leqslant_e \mathcal{U}'$ and $\mathcal{U} \neq \mathcal{U}'$. $\mathcal{U}$ is *maximal* if $\mathcal{U}$ is strong and there is no strong $\mathcal{U}'$ end-extending $\mathcal{U}$.

2.10 Theorem (3.9) *Assume* $\neg 0^{\mathrm{long}}$*. Let* $\mathcal{U}$ *be strong, and* $\gamma \in \mathrm{On}$*. Then*

i) $\mathcal{U} \restriction \gamma$ *is strong.*
ii) $K[\mathcal{U} \restriction \gamma] \subseteq K[\mathcal{U}]$*, indeed,* $K[\mathcal{U} \restriction \gamma] = (K[\bar{\mathcal{U}} \restriction \gamma])^{K[\mathcal{U}]}$*, where* $\bar{\mathcal{U}} = \mathcal{U} \cap K[\mathcal{U}]$*.*
iii) $\mathcal{P}(\gamma) \cap K[\mathcal{U} \restriction \gamma] = \mathcal{P}(\gamma) \cap K[\mathcal{U}]$*.*

2.11 Theorem (3.11) *Assume* $\neg 0^{\mathrm{long}}$*. Let* $\mathcal{U}$ *be strong. Then there is a maximal* $\mathcal{U}' \geqslant_e \mathcal{U}$*.*

2.12 Lemma (3.12) *Assume* $\neg 0^{\mathrm{long}}$*. Let* $\mathcal{U}$ *be strong. Then for every regular cardinal* η*,* $\sup(\mathrm{dom}(\mathcal{U}) \cap \eta) < \eta$*.*

2.13 Theorem (3.14) *Assume* $\neg 0^{\mathrm{long}}$*. Let* $K[\mathcal{U}]$*,* $K[\mathcal{U}']$ *be core models with* $\mathrm{dom}(\mathcal{U}) = \mathrm{dom}(\mathcal{U}')$*. Then* $|K[\mathcal{U}]| = |K[\mathcal{U}']|$ *and* $\mathcal{U} \cap K[\mathcal{U}] = \mathcal{U}' \cap K[\mathcal{U}']$*.*

2.14 Definition (3.15) Let $\mathcal{U}_{\mathrm{can}}$ be the unique maximal strong sequence satisfying

i) $\mathcal{U}_{\mathrm{can}} \subseteq K[\mathcal{U}_{\mathrm{can}}]$,
ii) if $\kappa \in \mathrm{dom}(\mathcal{U}_{\mathrm{can}})$, then κ is the minimal ordinal ζ such that there is some $\mathcal{U}' >_e \mathcal{U}_{\mathrm{can}} \restriction \kappa$ with $\zeta = \min \mathrm{dom}(\mathcal{U}' \setminus (\mathcal{U}_{\mathrm{can}} \restriction \kappa)$.

$\mathcal{U}_{\mathrm{can}}$ is called the *canonical sequence* and $K[\mathcal{U}_{\mathrm{can}}]$ is the *canonical core model*.

2.15 THEOREM (3.18) *Assume* $\neg 0^{\text{long}}$. *Let* $K[\mathcal{U}]$ *be a core model and let* $j : K[\mathcal{U}] \to K[\mathcal{U}]$ *be elementary with critical point* $\kappa > \sup\operatorname{dom}(\mathcal{U})$. *Assume* δ *is a regular cardinal greater than* κ *which is a limit cardinal in* $K[\mathcal{U}]$. *Then there exists a strong* $\mathcal{U}' >_e \mathcal{U}$ *with* $\tau := \min\operatorname{dom}(\mathcal{U}' \setminus \mathcal{U})$ *satisfying* $\tau \geqslant \kappa$ *and* $\tau = \delta$, *if* $\delta = \omega_1$, *and* $\tau < \delta$, *if* $\delta \geqslant \omega_2$.

2.16 THEOREM (3.20) *Assume* $\neg 0^{\text{long}}$. *Let* τ *be an ordinal such that* $\sup\operatorname{dom}(\mathcal{U}_{can} \restriction (\tau+1)) < \tau$. *Then*

i) *If* $\tau \geqslant \omega_2$ *is a limit ordinal and* $\operatorname{cf}(\tau) < \operatorname{card}(\tau)$, *then* τ *is singular in* $K[\mathcal{U}_{can}]$.

ii) *If* τ *is a singular cardinal in* V, *then* τ *is singular in* $K[\mathcal{U}_{can}]$ *and* $\tau^+ = \tau^{+K[\mathcal{U}_{can}]}$.

2.17 THEOREM (3.24) *Let* D *be a simple predicate with* $\operatorname{otp}\operatorname{dom}(D) \leqslant \min\operatorname{dom}(D)$ *or* $D = \emptyset$. *Let* Q *be a transitive model of a sufficiently large finite part of* $\mathrm{ZFC} + \mathrm{V} = K[\bar{D}]$, *where* $\bar{D} := D \cap Q \in Q$. *Let* $\omega_1 \subseteq Q$ *and* $\operatorname{dom}(D) = \operatorname{dom}(\bar{D})$. *Then*

i) *Let* M *be a* D*-mouse*, $\operatorname{meas}(M) \neq \emptyset$, *and let* $\kappa = \min\operatorname{meas}(M)$ *be singular in* Q. *Then* $\operatorname{lp}(M) \subseteq Q$.

ii) *Let* $\lambda \subseteq Q$ *be a cardinal greater than* $\sup\operatorname{dom}(D)$ *and assume the following condition is satisfied: If* $C \subseteq \lambda$ *is closed unbounded in* λ, *then there exists a* $\kappa \in C$ *which is singular in* Q. *Then* $K_\lambda[D] \subseteq Q$.

2.18 THEOREM (3.25) *Assume* $\neg 0^{\text{long}}$. *Let* $K[\mathcal{U}]$ *be a core model. Let* λ *be a cardinal greater than* $\gamma := \sup\operatorname{dom}(\mathcal{U})$. *Assume* $\pi : K_\lambda[\mathcal{U}] \to W$ *is elementary,* W *is transitive, and* π *has critical point* α, *greater than* γ. *Then there is an elementary embedding* $\bar{\pi} : K[\mathcal{U}] \to K[\mathcal{U}]$ *with critical point* α.

Chapter 3

Chang's Conjecture at club-many Points

HOTSPUR *Will this content you, Kate?*

LADY PERCY *It must of force.*

[Exeunt]

(William Shakespeare, King Henry IV, Part I, Act 2, Scene 3)

3.1 DEFINITION *Chang's Conjecture*, CC, is the statement that any structure $\mathfrak{A} = \langle A, P, \ldots \rangle$ of countable type such that $\mathrm{card}(A) = \omega_2$ and $\mathrm{card}(P) = \omega_1$ has an elementary substructure $\mathfrak{B} \prec \mathfrak{A}$, $\mathfrak{B} = \langle B, R, \ldots \rangle$, such that $\mathrm{card}(B) = \omega_1$ and $\mathrm{card}(R) = \omega$. This is also written as $\langle \omega_2, \omega_1 \rangle \twoheadrightarrow \langle \omega_1, \omega \rangle$. For regular cardinals $\kappa, \lambda, \mu, \nu$, $\langle \kappa, \lambda \rangle \twoheadrightarrow \langle \mu, \nu \rangle$ denotes the obvious generalisation. Instead of requiring $\mathrm{card}(R) = \nu$ one can also consider demanding $\mathrm{card}(R) < \nu$, denoted by $\langle \kappa, \lambda \rangle \twoheadrightarrow \langle \mu, {<}\nu \rangle$.

Silver [Sil71] showed that the existence of an ω_1-Erdős cardinal (cf. Definition 3.7) implies that CC holds in some forcing extension of the universe. He used a modification of the Levy collapse as partial order for his forcing, which is now aptly known as the Silver collapse. Later, Donder and Levinski [DL89] and Baumgartner [Bau91] showed that, in fact, the Levy collapse will also do. On the other hand, Donder [DJK81] established that if CC holds, then

ω_2 is ω_1-Erdős in an inner model, so that this is really the exact consistency strength of CC. We now want to consider what seems at first sight to be a strengthening of CC, requiring that there exist a closed unbounded set C such that the ordertype of the predicate R of $\mathfrak{B}$ can assume any value from this set. It will turn out that this is actually equivalent to the original CC.

Note first that by relabeling, if necessary, one can assume that $A = \omega_2$ and $P = \omega_1$ (or respectively κ and λ).

3.2 DEFINITION CC$^{\text{club}}$ is the statement that for any structure $\mathfrak{A} = \langle A, P, \ldots\rangle$ of countable type such that $A = \omega_2$ and $P = \omega_1$, there exists a closed unbounded set $C \subseteq \omega_1$ such that for all $\alpha \in C$ there exists an elementary substructure $\mathfrak{B}^\alpha \prec \mathfrak{A}$, $\mathfrak{B}^\alpha = \langle B^\alpha, R^\alpha, \ldots\rangle$, such that $\text{card}(B^\alpha) = \omega_1$ and $R^\alpha = \alpha$. We denote this by $\langle\omega_2, \omega_1\rangle \underset{\text{club}}{\twoheadrightarrow} \langle\omega_1, {<}\omega_1\rangle$.

For higher cardinals, one has to be slightly more careful, as the example in [Sch96, Lemma 6.2] shows that not all cofinalities for α are always possible. Thus, assuming $\nu = \varrho^+$, let $\langle\kappa, \lambda\rangle \underset{\text{club}}{\twoheadrightarrow} \langle\mu, {<}\nu\rangle$ denote the obvious generalization of the above statement, except that we now require C to be only ϱ-club in ν, and hence w.l.o.g. concentrating on α with $\text{cf}(\alpha) = \varrho$.

Note that if π_α denotes the inverse of the transitive collapse of $\mathfrak{B}^\alpha$ to $\mathfrak{C}^\alpha$, then α will be the critical point of π_α, and $\pi_\alpha(\alpha) = \lambda$.

It is known that CC has a combinatorial equivalent in terms of a partition property.

3.3 DEFINITION Let $\kappa \to [\mu]^{<\omega}_{\lambda,\nu}$ denote the statement that for any function $f : [\kappa]^{<\omega} \to \lambda$ there exists some $H \subseteq \kappa$, $\text{card}(H) = \mu$, such that $\text{card}(f''[H]^{<\omega}) = \nu$.

Let $\kappa \to [\mu]^{<\omega}_{\lambda,<\nu}$ denote the statement that for any function $f : [\kappa]^{<\omega} \to \lambda$ there exists some $H \subseteq \kappa$, $\text{card}(H) = \mu$, such that $\text{card}(f''[H]^{<\omega}) < \nu$.

3.4 LEMMA *Let $\kappa \geqslant \lambda$, $\kappa \geqslant \mu \geqslant \nu > \omega$. Then*

$$\langle\kappa, \lambda\rangle \twoheadrightarrow \langle\mu, {<}\nu\rangle \qquad \textit{iff} \qquad \kappa \to [\mu]^{<\omega}_{\lambda,<\nu}.$$

PROOF Cf. [Kan94, Theorem 8.1], or the proof of Lemma 3.6 below. ■

Note that $\langle \omega_2, \omega_1 \rangle \twoheadrightarrow \langle \omega_1, \omega \rangle$ is equivalent to $\langle \omega_2, \omega_1 \rangle \twoheadrightarrow \langle \omega_1, <\omega_1 \rangle$, and thus to $\omega_2 \rightarrow [\omega_1]^{<\omega}_{\omega_1, <\omega_1}$, as an infinite predicate can surely not be reduced to a finite one in a substructure.

It is thus natural to seek a combinatorial equivalent for $\mathrm{CC}^{\mathrm{club}}$:

3.5 DEFINITION Let $\kappa \underset{\text{club}}{\rightarrow} [\mu]^{<\omega}_{\lambda, <\nu}$ denote the statement that for any function $f : [\kappa]^{<\omega} \rightarrow \lambda$ which is onto there exists a ϱ-club set $C \subseteq \nu$, where $\nu = \varrho^+$, such that for any $\alpha \in C$ there is a set $H^\alpha \subseteq \kappa$, $\operatorname{card}(H^\alpha) = \mu$, such that $f''[H^\alpha]^{<\omega} = \alpha$.

3.6 LEMMA *Let $\kappa \geqslant \lambda$, $\kappa \geqslant \mu \geqslant \nu = \varrho^+ > \omega$. Then*

$$\langle \kappa, \lambda \rangle \underset{\text{club}}{\twoheadrightarrow} \langle \mu, <\nu \rangle \qquad \textit{iff} \qquad \kappa \underset{\text{club}}{\rightarrow} [\mu]^{<\omega}_{\lambda, <\nu}.$$

PROOF Assume $\langle \kappa, \lambda \rangle \underset{\text{club}}{\twoheadrightarrow} \langle \mu, <\nu \rangle$. Let $f : [\kappa]^{<\omega} \rightarrow \lambda$ be onto. Let $\mathfrak{A} := \langle \kappa, \lambda, \in, (f \restriction [\kappa]^n)_{n \in \omega} \rangle$. Let $C \subseteq \nu$ be a ϱ-club set guaranteed by the assumption. Choose $\alpha \in C$, and let $\mathfrak{B}^\alpha = \langle B^\alpha, R^\alpha, \in, \ldots \rangle$ be as guaranteed by the assumption. $\mathfrak{B}^\alpha$ is an elementary substructure of $\mathfrak{A}$, and $f''[\kappa]^{<\omega} = \lambda$, so that $f''[B^\alpha]^{<\omega} = \lambda \cap B^\alpha = R^\alpha = \alpha$, and $\operatorname{card}(B^\alpha) = \mu$.

On the other hand, assume that $\kappa \underset{\text{club}}{\rightarrow} [\mu]^{<\omega}_{\lambda, <\nu}$. Let $\mathfrak{A} = \langle \kappa, \lambda, \ldots \rangle$. Let $\{h_n \mid n \in \omega\}$ be a complete set of Skolem functions for $\mathfrak{A}$ such that h_n is $k(n)$-ary, $k(n) \leqslant n$. Define a function $f : [\kappa]^{<\omega} \rightarrow \lambda$ by setting

$$f(\xi_1, \ldots, \xi_n) := \begin{cases} h_n(\xi_1, \ldots, \xi_{k(n)}) & \text{if this is less than } \lambda, \\ 0 & \text{else.} \end{cases}$$

Note that $f''[\kappa]^{<\omega} = \lambda$, as e. g. $h_{\exists v_1 (v_1 = v_0)}(\xi) = \xi$, for $\xi < \lambda$. Thus we can apply the assumption to get some ϱ-club set $C \subseteq \nu$. Let $\alpha \in C$ and let $H^\alpha \subseteq \kappa$, $\operatorname{card}(H^\alpha) = \mu$, $f''[H^\alpha]^{<\omega} = \alpha$. Let $B^\alpha := \bigcup_{n \in \omega} h_n''[H^\alpha]^{k(n)}$, so that $\operatorname{card}(B^\alpha) = \mu$. Let $\mathfrak{B}^\alpha := \langle B^\alpha, \lambda \cap B^\alpha, \ldots \rangle$. Then $\mathfrak{B}^\alpha \prec \mathfrak{A}$. Also, $\lambda \cap B^\alpha \subseteq f''[H^\alpha]^{<\omega} = \alpha$. In fact, the converse is true, too: $f''[H^\alpha]^{<\omega}$ is a subset of λ by the definition of f, and a subset of B^α, by the definition of B^α. Thus $R^\alpha = \alpha$. ■

3.7 DEFINITION $\beta \to (\alpha)^{<\omega}_{\delta}$ denotes the statement that for any $f : [\beta]^{<\omega} \to \delta$ there exsits an $H \subseteq \beta$ with $\operatorname{card}(H) = \alpha$ such that H is homogeneous for f, i. e., $\operatorname{card}(f''[H]^n) \leqslant 1$ for every $n \in \omega$. For $\alpha > \omega$, any λ that satisfies $\lambda \to (\alpha)^{<\omega}_2$ is called *α-Erdős*, and the least such λ is denoted by $\kappa(\alpha)$.

3.8 THEOREM *Suppose $\kappa = \kappa(\omega_1)$, the least ω_1-Erdős cardinal. Then*

$$\Vdash_{\operatorname{Col}(\omega_1,\kappa)} \kappa = \dot{\omega}_2 \ \wedge\ \mathrm{CC}^{\mathrm{club}}.$$

PROOF The proof is based on that of that of [Kan00, Theorem 33.10]. Let $\kappa = \kappa(\omega_1)$ and let $P := \operatorname{Col}(\omega_1, \kappa)$ be the Levy collapse, i. e.,

$$P := \{p \mid \operatorname{fun}(p) \ \wedge\ \operatorname{dom}(p) \subseteq \kappa \times \omega_1 \ \wedge\ \operatorname{card}(p) < \omega_1 \ \wedge \\ \forall \langle \alpha, \xi\rangle \in \operatorname{dom}(p)\ (\alpha > 0 \ \to\ p(\alpha,\xi) < \alpha)\}.$$

Then P is ω_1-closed, and as κ is inaccessible, P has the κ-chain condition (κ-c.c.). P collapses κ to ω_2, preserving ω_1 and cardinals greater than κ (cf. [Kan94, §10]). Assume that $p_0 \in P$ is a forcing condition such that

$$p_0 \Vdash \dot{f} : [\kappa]^{<\omega} \to \omega_1 \text{ onto}.$$

Let G be a P-generic filter over V. We now seek a closed unbounded set $C \subseteq \omega_1$ such that, in $\mathrm{V}[G]$,

$$\forall \alpha \in C\ \exists Y^\alpha \subseteq \kappa\ (\operatorname{card}(Y^\alpha) = \omega_1 \ \wedge\ \dot{f}^{G}{}''[Y^\alpha]^{<\omega} = \alpha).$$

We will actually find such a set C independently of G. Note that if $C \in \mathrm{V}$ is club in V, then C is club in $\mathrm{V}[G]$, too.

Since κ is the least ω_1-Erdős cardinal, for each $\gamma < \kappa$ there exists a function $f_\gamma : [\gamma]^{<\omega} \to 2$ witnessing $\gamma \not\to (\omega_1)^{<\omega}_2$. For $n \in \omega$, let $g_n : [\kappa]^{n+1} \to 2$ be defined by

$$g_n(\alpha_1, \ldots, \alpha_n, \gamma) := f_\gamma(\alpha_1, \ldots, \alpha_n).$$

Let $W : \mathrm{V}_\kappa \xrightarrow{\sim} \kappa$ be a bijection (recall that κ is inaccessible), such that for all $\delta < \kappa$ such that δ is inaccessible, too, $W \restriction \delta : \mathrm{V}_\delta \xrightarrow{\sim} \delta$ is also bijective. Finally, code $\dot{f}$ into a predicate $R \subseteq P \times [\kappa]^{<\omega} \times \omega_1$ by setting

$$\langle p, x, \beta\rangle \in R \ \leftrightarrow\ p \leqslant p_0 \ \wedge\ p \Vdash \dot{f}(\check{x}) = \beta.$$

Let $\mathfrak{A} := \langle \mathrm{V}_\kappa, \in, \omega_1, \{\omega_1\}, P, \leqslant_P, \{p_0\}, W, R, (g_n)_{n\in\omega}\rangle$, and let $\langle h_n \mid n \in \omega\rangle$ be a complete set of definable (using W) Skolem functions for $\mathfrak{A}$. Assume w.l.o.g. that h_n is $k(n)$-ary with $k(n) \leqslant n$. Let $\langle \varphi_\xi \mid \xi < \omega_1\rangle$ be an enumeration of the formulae of $\mathcal{L}_{\mathfrak{A}} \cup \{c_\beta \mid \beta < \omega_1\}$. Define a function $g : [\kappa]^{<\omega} \to \omega_1$ by setting

$$g(\xi_1, \dots, \xi_n) := \begin{cases} h_n(\xi_1, \dots, \xi_{k(n)}) & \\ & \text{if } m = 2n+1 \text{ and } h_n(\xi_1, \dots, \xi_{k(n)}) < \omega_1, \\ 0 & \text{if } m = 2n+1 \text{ and } h_n(\xi_1, \dots, \xi_{k(n)}) \geqslant \omega_1, \\ 0 & \text{if } m = 2n \text{ and } \langle \xi_1, \dots, \xi_n\rangle \text{ and } \langle \xi_{n+1}, \dots, \xi_{2n}\rangle \\ & \quad \text{realize the same type over } \langle \mathfrak{A}, (c_\beta)_{\beta<\omega_1}\rangle, \\ \xi & \text{if } m = 2n \text{ and } \varphi_\xi \text{ is a witness to the fact that} \\ & \quad \langle \xi_1, \dots, \xi_n\rangle \text{ and } \langle \xi_{n+1}, \dots, \xi_{2n}\rangle \text{ do not} \\ & \quad \text{realize the same type over } \langle \mathfrak{A}, (c_\beta)_{\beta<\omega_1}\rangle. \end{cases}$$

As κ is ω_1-Erdős, let $E \subseteq \kappa$, $\mathrm{card}(E) = \omega_1$, be homogeneous for g. Let $E = \{\eta_\zeta \mid \zeta < \omega_1\}$ be the increasing enumeration and assume w.l.o.g. that η_ω is minimal. Let

$$H := H_0 := \{h_n(\zeta_1, \dots, \zeta_{k(n)}) \mid n \in \omega \ \wedge\ \zeta_1 < \dots < \zeta_{k(n)} \in E\}$$
$$H_\alpha := \{h_n(\zeta_1, \dots, \zeta_{k(n)}) \mid n \in \omega \ \wedge\ \zeta_1 < \dots < \zeta_{k(n)} \in E \cup \alpha\},$$

and let $\mathfrak{H}$, $\mathfrak{H}_\alpha$ be the corresponding substructures of respectively $\mathfrak{A}$ and $\langle \mathfrak{A}, (c_\beta)_{\beta<\alpha}\rangle$. To enhance legibility, for any $\alpha \leqslant \omega_1$, let $\mathfrak{A}_\alpha$ denote $\langle \mathfrak{A}, (c_\beta)_{\beta<\alpha}\rangle$.

Claim 1 For any $\alpha \in \omega_1$, E is a set of indiscernibles for $\mathfrak{H}_\alpha$.

Proof Let $\alpha \in \omega_1$, and $\langle \xi_1, \dots, \xi_n\rangle$ and $\langle \zeta_1, \dots, \zeta_n\rangle$ from $[E]^{<\omega}$. Choose another n-tuple $\langle \vartheta_1, \dots, \vartheta_n\rangle$ from $[E]^n$, such that $\vartheta_1 > \max(\xi_n, \zeta_n)$. E is homogeneous for g, so

$$g(\xi_1, \dots, \xi_n, \vartheta_1, \dots, \vartheta_n) = g(\zeta_1, \dots, \zeta_n, \vartheta_1, \dots, \vartheta_n) = \xi,$$

for some $\xi \in \omega_1$. If $\xi = 0$, then $\langle \xi_1, \dots, \xi_n\rangle$ and $\langle \zeta_1, \dots, \zeta_n\rangle$ realize the same type over $\mathfrak{A}_{\omega_1}$ and so, a fortiori, over $\mathfrak{A}_\alpha$, so that

$$\mathfrak{H}_\alpha \vDash \varphi(\xi_1, \dots, \xi_n) \ \leftrightarrow\ \mathfrak{H}_\alpha \vDash \varphi(\zeta_1, \dots, \zeta_n).$$

If, on the other hand, $\xi > 0$, then find another n-tuple $\langle \iota_1, \dots, \iota_n \rangle$ from $[E]^n$, such that $\iota_1 > \vartheta_n$. Then

$$g(\xi_1, \dots, \xi_n, \iota_1, \dots, \iota_n) = g(\vartheta_1, \dots, \vartheta_n, \iota_1, \dots, \iota_n) = \xi,$$

too. But this gives the following absurdity:

$$\begin{aligned} &\mathfrak{A}_{\omega_1} \vDash \varphi_\xi(\xi_1, \dots, \xi_n) \;\leftrightarrow \\ &\qquad \neg\mathfrak{A}_{\omega_1} \vDash \varphi_\xi(\vartheta_1, \dots, \vartheta_n) \;\leftrightarrow\; \mathfrak{A}_{\omega_1} \vDash \varphi_\xi(\iota_1, \dots, \iota_n) \\ &\qquad\qquad \leftrightarrow\; \neg\mathfrak{A}_{\omega_1} \vDash \varphi_\xi(\xi_1, \dots, \xi_n). \end{aligned}$$

Note that any $\eta \in E$ must be at least ω_1, as else $\mathfrak{A}_{\eta+1} \vDash c_\eta = \eta$, which no other element $\zeta \in E$ can possibly satisfy. □(Claim 1)

Claim 2 For any $\alpha \in \omega_1$, $H_\alpha \cap \omega_1$ is countable.

Proof Let $\alpha \in \omega_1$, and $\beta_1, \dots, \beta_l \in \alpha$, $\xi_1, \dots, \xi_n \in E$. Let h be one of the Skolem functions and assume that $h(\beta_1, \dots, \beta_l, \xi_1, \dots, \xi_n) < \omega_1$. If $l = 0$, then by the homogeneity of E for g, we get

$$h(\xi_1, \dots, \xi_n) = h(\zeta_1, \dots, \zeta_n)$$

for any n-tuple $\langle \zeta_1, \dots, \zeta_n \rangle$ from E. Thus for $l = 0$, only countably many different values are possible. If $l > 0$, then let $\gamma := h(\beta_1, \dots, \beta_l, \xi_1, \dots, \xi_n) < \omega_1$. Assume w.l.o.g. that $\alpha > \gamma$. If $\langle \zeta_1, \dots, \zeta_n \rangle$ is another n-tuple from E, then by Claim 1 (and as the Skolem functions are definable) we may conclude that

$$\begin{aligned} &\mathfrak{A} \vDash h(\beta_1, \dots, \beta_l, \xi_1, \dots, \xi_n) = \gamma \\ &\;\leftrightarrow\; \mathfrak{A}_\alpha \vDash h(c_{\beta_1}, \dots, c_{\beta_l}, \xi_1, \dots, \xi_n) = c_\gamma \\ &\;\leftrightarrow\; \mathfrak{A}_\alpha \vDash h(c_{\beta_1}, \dots, c_{\beta_l}, \zeta_1, \dots, \zeta_n) = c_\gamma \\ &\;\leftrightarrow\; \mathfrak{A} \vDash h(\beta_1, \dots, \beta_l, \zeta_1, \dots, \zeta_n) = \gamma. \end{aligned}$$

Now $[\alpha]^{<\omega}$ is countable and there are only countably many Skolem functions, so that, indeed, $H_\alpha \cap \omega_1$ is countable. □(Claim 2)

Claim 3 The set $C := \{\alpha \in \omega_1 \mid H_\alpha \cap \omega_1 = \alpha\}$ is closed and unbounded in ω_1.

PROOF To see that C is closed, let $\langle \alpha_i \mid i < \gamma \rangle$ be a sequence from C, $\gamma < \omega_1$. Let $\alpha := \bigcup_{i<\gamma} \alpha_i$. Let $\zeta \in H_\alpha \cap \omega_1$. Then there are some Skolem function h, some $\beta_1, \ldots, \beta_l \in \alpha$, and some $\zeta_1, \ldots, \zeta_n \in E$ such that

$$\zeta = h(\beta_1, \ldots, \beta_l, \zeta_1, \ldots, \zeta_n).$$

Obviously, there exists some $i_0 < \gamma$ such that $\beta_1, \ldots, \beta_l \in \alpha_{i_0}$, whence

$$\zeta \in H_{\alpha_{i_0}} \cap \omega_1 = \alpha_{i_0} \subseteq \alpha.$$

Thus $H_\alpha \cap \omega_1 \subseteq \alpha$, and the converse is trivially true, so that $\alpha \in C$.

To see that C is unbounded in ω_1, let $\alpha_0 < \omega_1$ be arbitrary. Define inductively $\alpha_{i+1} := \sup(H_{\alpha_i} \cap \omega_1) < \omega_1$ for $i < \omega$ and set $\alpha := \bigcup_{i<\omega} \alpha_i$. Again $\alpha \subseteq H_\alpha \cap \omega_1$ is trivially true. Let $\zeta \in H_\alpha \cap \omega_1$. Then there are some Skolem function h, some $\beta_1, \ldots, \beta_l \in \alpha$, and some $\zeta_1, \ldots, \zeta_n \in E$ such that

$$\zeta = h(\beta_1, \ldots, \beta_l, \zeta_1, \ldots, \zeta_n).$$

Again, there exists some $i_0 < \omega$ such that $\beta_1, \ldots, \beta_l \in \alpha_{i_0}$, so that now

$$\zeta \in H_{\alpha_{i_0}} \cap \omega_1 \subseteq \alpha_{i_0+1} \subseteq \alpha.$$

$\Box$(Claim 3)

CLAIM 4 For any $\alpha \in \omega_1$ the set E is *α-remarkable*, i. e., for any $(l+m+n)$-ary Skolem function h, any $\beta_1 < \ldots < \beta_l < \alpha$, and any $\zeta_1 < \ldots < \zeta_{m+n} < \omega_1$,

$$\text{if } h(\beta_1, \ldots, \beta_l, \eta_{\zeta_1}, \ldots, \eta_{\zeta_{m+n}}) \in \eta_{\zeta_{m+1}}, \text{ then for all } \zeta_m < \delta_1 < \ldots < \delta_n < \omega_1$$
$$h(\beta_1, \ldots, \beta_l, \eta_{\zeta_1}, \ldots, \eta_{\zeta_{m+n}}) = h(\beta_1, \ldots, \beta_l, \eta_{\zeta_1}, \ldots, \eta_{\zeta_m}, \eta_{\delta_1}, \ldots, \eta_{\delta_n}).$$

PROOF Assume that for some $\alpha \in \omega_1$, some Skolem function h, some $\beta_1 < \ldots < \beta_l < \alpha$, and some $\zeta_1 < \ldots < \zeta_{m+n} < \omega_1$,

$$h(\beta_1, \ldots, \beta_l, \eta_{\zeta_1}, \ldots, \eta_{\zeta_{m+n}}) \in \eta_{\zeta_{m+1}},$$

but

$$h(\beta_1, \ldots, \beta_l, \eta_{\zeta_1}, \ldots, \eta_{\zeta_{m+n}}) \neq h(\beta_1, \ldots, \beta_l, \eta_{\zeta_1}, \ldots, \eta_{\zeta_m}, \eta_{\delta_1}, \ldots, \eta_{\delta_n})$$

for some (and hence by indiscernibility for all) n-tuple $\delta_1, \ldots, \delta_n$ from ω_1 such that $\delta_1 > \zeta_m$. Now partition E into successive bits s_γ of length n, with the exception of s_0, having length m. I. e., $s_0 = \{\eta_1, \ldots, \eta_m\}$, $s_1 = \{\eta_{m+1}, \ldots, \eta_{m+n}\}$, and $s_\omega = \{\eta_\omega, \ldots, \eta_{\omega+n-1}\}$. By indiscernibility, for $\gamma < \delta < \omega_1$,

$$h(\vec{\beta}, s_0, s_\gamma) \neq h(\vec{\beta}, s_0, s_\delta).$$

But surely $h(\vec{\beta}, s_0, s_\gamma) > h(\vec{\beta}, s_0, s_\delta)$ is impossible, as this would give an infinite descending chain of ordinals. Thus $\gamma < \delta$ must imply $h(\vec{\beta}, s_0, s_\gamma) < h(\vec{\beta}, s_0, s_\delta)$. Note that by assumption, and indiscernibility, we have

$$h(\vec{\beta}, s_0, s_\omega) \in \eta_\omega.$$

But $\{h(\vec{\beta}, s_0, s_\gamma) \mid \gamma < \omega_1\}$ is a set of indiscernibles for $\mathfrak{A}$ of size ω_1, and hence homogeneous for g, contradicting the minimality of η_ω. $\square$(Claim 4)

Claim 5 For any $\alpha \in \omega_1$, Condition II of the theory of $0^\#$ holds for $E \cup \alpha$, i. e., for any $(l+n)$-ary Skolem function h, any $\beta_1 < \ldots < \beta_l < \alpha$, and any $\zeta_1 < \ldots < \zeta_{n+1} < \omega_1$,

$$\text{if } h(\beta_1, \ldots, \beta_l, \eta_{\zeta_1}, \ldots, \eta_{\zeta_n}) \in \mathrm{On} \text{ then } h(\beta_1, \ldots, \beta_l, \eta_{\zeta_1}, \ldots, \eta_{\zeta_n}) \in \eta_{\zeta_{n+1}}.$$

Proof If the claim is false, then using indiscernibility one sees that

$$E' := E \setminus \eta_{\zeta_{n+1}} \subseteq \gamma := h(\beta_1, \ldots, \beta_l, \eta_{\zeta_1}, \ldots, \eta_{\zeta_n}) + 1.$$

E' has cardinality ω_1, is a subset of γ, and the functions g_n ensure that it is homogeneous for f_γ: let $m \in \omega$, $\iota \in \{0, 1\}$, and $\langle \eta_{\vartheta_1}, \ldots, \eta_{\vartheta_m} \rangle, \langle \eta_{\kappa_1}, \ldots, \eta_{\kappa_m} \rangle \in [E']^m$. Then

$$\begin{aligned}
& f_\gamma(\eta_{\vartheta_1}, \ldots, \eta_{\vartheta_m}) = \iota \\
& \leftrightarrow \mathfrak{A} \models g_n(\eta_{\vartheta_1}, \ldots, \eta_{\vartheta_m}, \gamma) = \iota \\
& \leftrightarrow \mathfrak{A} \models \varphi(\iota, \beta_1, \ldots, \beta_l, \eta_{\zeta_1}, \ldots, \eta_{\zeta_n}, \eta_{\vartheta_1}, \ldots, \eta_{\vartheta_m}) \\
& \leftrightarrow \mathfrak{A}_\alpha \models \varphi(\iota, c_{\beta_1}, \ldots, c_{\beta_l}, \eta_{\zeta_1}, \ldots, \eta_{\zeta_n}, \eta_{\vartheta_1}, \ldots, \eta_{\vartheta_m}) \\
& \leftrightarrow \mathfrak{A}_\alpha \models \varphi(\iota, c_{\beta_1}, \ldots, c_{\beta_l}, \eta_{\zeta_1}, \ldots, \eta_{\zeta_n}, \eta_{\kappa_1}, \ldots, \eta_{\kappa_m}) \\
& \leftrightarrow f_\gamma(\eta_{\kappa_1}, \ldots, \eta_{\kappa_m}) = \iota.
\end{aligned}$$

But this contradicts the choice of f_γ as a witness to $\gamma \not\to (\omega_1)^{<\omega}_2$. $\square$(Claim 5)

Claim 6 If $\eta\in E$, then η is inaccessible.

Proof By indiscernibility, it suffices to show that η_0 is inaccessible. So let j be an order-preserving injection from E to E with $j(\eta_0) > \eta_0$. This map easily extends to an elementary embedding $\tilde{\jmath} : \mathfrak{H} \to \mathfrak{H}$. By remarkability, we must have $\mathrm{crit}(\tilde{\jmath}) = \eta_0$: Let $\gamma = h(\eta_{\zeta_1}, \dots, \eta_{\zeta_n}) < \eta_0$. Remarkability implies that we can substitute for $\eta_{\zeta_1}, \dots, \eta_{\zeta_n}$ any other n-tuple of indiscernibles (we have $\alpha = m = 0$), so in particular

$$\gamma = h(\eta_{\zeta_1}, \dots, \eta_{\zeta_n}) = h(j(\eta_{\zeta_1}), \dots, j(\eta_{\zeta_n})) = \tilde{\jmath}(\gamma).$$

From this, we can conclude that η_0 is totally indescribable, whence inaccessible: Assume that for some formula φ

$$\mathfrak{H} \models \Big(R \subseteq V_{\eta_0} \ \wedge\ \langle V_{\eta_0},\in, R\rangle \models \varphi\Big).$$

Then

$$\mathfrak{H} \models \exists\alpha < \tilde{\jmath}(\eta_0)\ \Big(R \subseteq V_\alpha \ \wedge\ \langle V_\alpha,\in, R\rangle \models \varphi\Big),$$

and since $\tilde{\jmath}$ is an elementary embedding from $\mathfrak{H}$ to $\mathfrak{H}$,

$$\mathfrak{H} \models \exists\alpha < \eta_0\ \Big(R \subseteq V_\alpha \ \wedge\ \langle V_\alpha,\in, R\rangle \models \varphi\Big),$$

so η_0 is totally indescribable (and thus inaccessible) in $\mathfrak{H}$. Since $\mathfrak{H} \prec \mathfrak{A}$, η_0 is really inaccessible. $\square$(Claim 6)

Consider now the partial order P. Let $P_\gamma := \mathrm{Col}(\omega_1, \gamma)$. Then $P \simeq P_\gamma \times \mathrm{Col}(\omega_1, \kappa \setminus \gamma)$, by [Kan94, Lemma 10.17 b)].

Claim 7 If A is a maximal antichain in P_γ, then it is a maximal antichain in P.

Proof Let A be a maximal antichain in P_γ. Let $p\in P \setminus A$. Then $p\restriction\gamma\in P_\gamma$, so that there is some $q\in A$ such that $p\restriction\gamma$ and q are compatible. But surely $p\restriction(\kappa\setminus\gamma)$ and q are compatible, too, so that in the end p and q are compatible. Thus A is maximal in P, too. $\square$(Claim 7)

Let $S := \{x\in[E]^{<\omega_1}\setminus\{0\} \mid x \text{ has no last element}\}$, and define a partial order on S by setting

$$x <_i y \ \leftrightarrow\ \exists\gamma < \cup y\ (x = y \cap \gamma),$$

i. e., $x <_i y$ iff x is a proper initial segment of y.

From now on, let $\alpha \in C$. For $x \in S$, let N_x^α be the domain of the Skolem hull of $x \cup \alpha$ in $\mathfrak{A}$. Then N_x^α is a countable model of ZFC.

For $p \in P$, say that p is *P-generic over* N_x^α iff $\{q \in P \cap N_x^\alpha \mid q \geqslant p\}$ is P-generic over N_x^α, and similarly define *P_γ-generic* for $\gamma \in N_x^\alpha \cap \kappa$.

Claim 8 Let $x, y \in S$, $x <_i y$, $\gamma := \min(y \setminus x)$. Assume that p is P-generic over N_x^α. Then p is P_γ-generic over N_y^α.

Proof Let $A \in N_y^\alpha$ be a maximal antichain in P_γ. Then

$$A = h(\beta_1, \dots, \beta_n, \eta_{\zeta_1}, \dots, \eta_{\zeta_m}, \eta_{\zeta_{m+1}}, \dots, \eta_{\zeta_{m+n}}),$$

for some $\beta_1, \dots, \beta_n \in \alpha$, $\eta_{\zeta_1}, \dots, \eta_{\zeta_m} \in x$, and $\eta_{\zeta_{m+1}}, \dots, \eta_{\zeta_{m+n}} \in y \setminus x$. By Claim 6, γ is inaccessible, so P_γ has the γ-c.c. Thus $\operatorname{card}(A) < \gamma$, and hence $A \in V_\gamma$. Recall that $W \restriction \mathrm{V}_\gamma : \mathrm{V}_\gamma \xrightarrow{\sim} \gamma$ is bijective. Applying α-remarkability, one can find $\eta_{\delta_1}, \dots, \eta_{\delta_n} \in x$ (!) such that $A = h(\beta_1, \dots, \beta_n, \eta_{\zeta_1}, \dots, \eta_{\zeta_m}, \eta_{\delta_1}, \dots, \eta_{\delta_n})$. This shows that $A \in N_x^\alpha$. Since A is a maximal antichain in P_γ, it is also a maximal antichain in P, by Claim 7. Since p was assumed to be P-generic over N_x^α, there must exist some $q \in P \cap N_x^\alpha$ such that $q \geqslant p$ and $q \in A$. But $N_x^\alpha \subseteq N_y^\alpha$, so that $\{q \in P_\gamma \cap N_y^\alpha \mid q \geqslant p\} \cap A \neq \emptyset$. Thus p is P_γ-generic over N_y^α. □(Claim 8)

Claim 9 Suppose $x \in S$, and p is P-generic over N_x^α. Then there exists some $y \in S$, $y >_i x$, and some $p' \leqslant p$ such that p' is P-generic over N_y^α.

Proof Let $\tau := \sup\{\alpha < \sup(E) \mid \exists \xi\ (\langle \alpha, \xi \rangle \in \operatorname{dom}(p))\}$. Then $\tau < \sup(E)$, as p is countable. Choose some $y \in S$, $y >_i x$, such that $y \setminus x \subseteq E \setminus (\tau + 1)$. Set $\gamma := \min(y \setminus x)$. By Claim 8,

$$G_0 := \{q \in P_\gamma \cap N_y^\alpha \mid q \geqslant p\}$$

is P_γ-generic over N_y^α. Apply the product analysis ([Kan94, Lemma 10.17 b)]) to $P \simeq P_\gamma \times \operatorname{Col}(\omega_1, \kappa \setminus \gamma)$ to find some G_1 such that $G_0 \times G_1$ is P-generic over N_y^α. (Recall that N_y^α is countable, so that generics exist.) Note that $\mathrm{On} \cap N_y^\alpha \subseteq \sup(E)$, by Claim 5, so that if $r \in G_1$, then $\operatorname{dom}(r) \subseteq \big(\sup(E) \setminus (\tau + 1)\big) \times \omega_1$. Thus if $r \in G_1$, then p and r are compatible, since

$\mathrm{dom}(p) \subseteq \big((\tau+1) \cup (\kappa \setminus \sup(E))\big) \times \omega_1$. Finally, set $p' := p \cup \bigcup G_1$. Then $p' \in P$, as G_1 is countable and P is ω_1-closed, and p' is P-generic over N^α_y. $\square$(Claim 9)

We will now proceed to show that whenever G is P-generic over V such that $p_0 \in G$, then there exists some set $Y^\alpha \in \mathrm{V}[G]$ of size ω_1 such that $\dot{f}^{G\prime\prime}[Y^\alpha]^{<\omega} = \alpha$.

Define recursively in $\mathrm{V}[G]$ a sequence $\langle y_\xi \mid \xi < \omega_1 \rangle$ of elements of S, such that $\xi < \zeta$ implies $y_\xi <_i y_\zeta$, and a sequence $\langle q_\xi \mid \xi < \omega_1 \rangle$ of elements of G, such that $q_\xi \leqslant p_0$ and such that q_ξ is P-generic over $N^\alpha_{y_\xi}$.

Set $y_0 := \{\eta_n \mid n \in \omega\}$, the first ω-many elements of E. Note that $p_0 \in N^\alpha_y$, as it was included as a constant in $\mathfrak{A}$. Now find some $q_0 \in G$, $q_0 \leqslant p_0$, such that q_0 is P-generic over $N^\alpha_{y_0}$. To this end, let $\langle D_n \mid n \in \omega \rangle$ be an enumeration of the dense subsets of P from $N^\alpha_{y_0}$. Note that if $N^\alpha_{y_0} \vDash D$ dense in P, then by elementarity, $\mathfrak{A} \vDash D$ dense in P, too. So inductively find $r_0 \leqslant p_0$ and $r_n \leqslant r_{n-1}$ such that $r_n \in G \cap D_n$. As P is ω_1-closed, there exists $q_0 \in G$ such that $\forall n \in \omega (q_0 \leqslant r_n)$. Then q_0 is as desired.

At successor steps, assume y_ξ and q_ξ to be given. Then the set $\{q \leqslant q_\xi \mid q \leqslant p_0 \ \wedge\ \exists y\ (y >_i y_\xi \ \wedge\ q \text{ is } P\text{-generic over } N^\alpha_y)\}$ is dense in P below q_ξ: For let $r \leqslant q_\xi$. Then r is P-generic over $N^\alpha_{y_\xi}$, too, and by Claim 9 there exists some $y_r >_i y_\xi$ and some $q_r \leqslant r$ such that q_r is P-generic over $N^\alpha_{y_r}$. But now, since G is P-generic and $q_\xi \in G$, there must exist some $q_{\xi+1} \leqslant q_\xi$, $q_{\xi+1} \in G \cap \{q \leqslant q_\xi \mid q \leqslant p_0 \ \wedge\ \exists y\ (y >_i y_\xi \ \wedge\ q \text{ is } P\text{-generic over } N^\alpha_y)\}$. Let $y_{\xi+1}$ be an appropriate witness.

At limit stages $\delta < \omega_1$, set $y_\delta := \bigcup_{\xi<\delta} y_\xi$ and, using the ω_1-closure of P and the genericity of G, find some $q_\delta \in G$ such that for all $\xi < \delta$, $q_\delta \leqslant q_\xi$. Then obviously q_δ is P-generic over $N^\alpha_{y_\delta}$.

Finally, set $Y^\alpha := \bigcup_{\xi<\omega_1} N^\alpha_{y_\xi} \cap \kappa$. All N^α_y are countable, so $\mathrm{card}(Y^\alpha) \leqslant \omega_1$, and as $y_\xi <_i y_{\xi+1}$, in fact, one has $\mathrm{card}(Y^\alpha) = \omega_1$. Now

$$\dot{f}^{G\prime\prime}[Y^\alpha]^{<\omega} \subseteq \omega_1 \cap \bigcup_{\xi<\omega_1} N^\alpha_{y_\xi} \subseteq \omega_1 \cap H^\alpha = \alpha,$$

for if $\langle \gamma_1, \ldots, \gamma_n \rangle \in [Y^\alpha]^n$, then $\langle \gamma_1, \ldots, \gamma_n \rangle \in \bigcup_{\xi<\delta} N^\alpha_{y_\xi} = N^\alpha_{y_\delta}$, for some $\delta < \omega_1$, so that $\dot{f}^G(\gamma_1, \ldots, \gamma_n) \in N^\alpha_{y_\delta}$, too. But the converse is true, too. Let $\beta < \alpha$.

Then $\beta \in N^\alpha_{y_0}$, and $D_\beta := \{q \leqslant p_0 \mid \exists x \in [\kappa]^{<\omega} \ (q \Vdash \dot{f}(\check{x}) = \beta)\}$ is an element of $N^\alpha_{y_0}$ and dense in P below p_0 (as $p_0 \Vdash \dot{f}$ onto). Now q_0 is P-generic over $N^\alpha_{y_0}$, and $q_0 \in G$. So there is indeed some $x \in [\kappa]^{<\omega} \cap N^\alpha_{y_0}$ such that $\dot{f}^G(\check{x}^G) = \beta$. But then $x \in [Y^\alpha]^{<\omega}$, as $N^\alpha_{y_0} \cap \kappa \subseteq Y^\alpha$. Thus $\beta \in \dot{f}^{G\prime\prime}[Y^\alpha]^{<\omega}$.

Thus $\dot{f}^{G\prime\prime}[Y^\alpha]^{<\omega} = \alpha$, as desired.

We have shown that given any p_0 such that $p_0 \Vdash \dot{f} : [\kappa]^{<\omega} \to \omega_1$ onto, there exists some closed unbounded set $C \subseteq \omega_1$ such that for any $\alpha \in C$ and for any P-generic filter G with $p_0 \in G$, there is a set $Y^\alpha \subseteq \kappa$, $\mathrm{card}(Y^\alpha) = \omega_1$, such that $\mathrm{V}[G] \vDash \dot{f}^{G\prime\prime}[Y^\alpha]^{<\omega} = \alpha$. That is, $\Vdash_{\mathrm{Col}(\omega_1,\kappa)} \kappa = \dot{\omega}_2 \ \wedge \ \mathrm{CC}^{\mathrm{club}}$. ■

Note that the proof will only work for $\mathrm{CC}^{\mathrm{club}}$ and not for higher transfer properties, say $\langle \kappa^+, \kappa \rangle \underset{\text{club}}{\twoheadrightarrow} \langle \kappa, < \nu \rangle$, where $\kappa = \nu^+ > \omega_1$: The countability of the N^α_x is essential to get the required generic objects G in the proof of Claim 9. This is no coincidence. For example, [Sch96, Theorem 6.2] shows that one needs at least a strong cardinal to get a model for $\langle \aleph_4, \aleph_3 \rangle \twoheadrightarrow \langle \aleph_3, \aleph_2 \rangle$ and $2^{\aleph_1} = \aleph_2$. So far, any forcing to get this uses, in fact, a huge cardinal [For82].

Chapter 4

The Transversal Hypothesis

> *"Contrariwise," continued Tweedledee, "if it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic."*
>
> *(Lewis Carroll, Through the Looking-Glass, Chapter IV)*

We will now consider a combinatorial principle closely related to Chang's Conjecture.

4.1 Definition Let κ be a regular cardinal. The *Transversal Hypothesis*, $\mathrm{TH}(\kappa)$, is the statement: There exists a sequence of functions $\langle g_\xi \mid \xi < \kappa^+\rangle$ such that for all ξ, g_ξ is regressive, and that for all $\xi < \zeta$, $\{\alpha < \kappa \mid g_\xi(\alpha) = g_\zeta(\alpha)\}$ has cardinality less than κ. $\mathrm{TH}^*(\kappa)$ is the same statement, with "has cardinality less than κ" replaced by "is not stationary". In this case, the set $\{\alpha < \kappa \mid g_\xi(\alpha) \neq g_\zeta(\alpha)\}$ contains a club set.

Assume $\kappa = \varrho^+$. Let $\mathrm{TH}^{\mathrm{stat}}(\kappa)$ be the statement: There exists a set $S \subseteq \kappa$, a sequence of functions $\langle g_\xi \mid \xi < \kappa^+\rangle$ and sequences of sets $\langle D_\xi \mid \xi < \kappa\rangle$ and $\langle D_{\xi\zeta} \mid \xi < \zeta < \kappa\rangle$ such that

i) S is stationary and $S \subseteq \mathrm{CF}_\varrho := \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \varrho\}$.

ii) $\forall \xi < \kappa^+ \Big(D_\xi$ is club in κ, $D_\xi \cap S \subseteq \mathrm{dom}(g_\xi)$, and g_ξ is regressive$\Big)$.

iii) $\forall \xi < \zeta < \kappa^+ \Big(D_{\xi\zeta}$ is club in κ and

$$D_{\xi\zeta} \cap S \subseteq \{\alpha \in D_\xi \cap D_\zeta \cap S \mid g_\xi(\alpha) \neq g_\zeta(\alpha)\}\Big).$$

Obviously, $\mathrm{TH}(\kappa) \to \mathrm{TH}^*(\kappa) \to \mathrm{TH}^{\mathrm{stat}}(\kappa)$. Jensen observed (cf. [Don84]) that $\mathrm{TH}^*(\kappa) \to \mathrm{TH}(\kappa)$. It suffices that the g_ξ only be defined on club sets themselves. $\mathrm{TH}^{\mathrm{stat}}(\kappa)$ intends to capture the essence of $\mathrm{TH}^*(\kappa)$ when restricted to a fixed stationary set S. Shelah [She86] has shown that without the requirement that $S \subseteq \mathrm{CF}_\varrho$, $\mathrm{TH}^{\mathrm{stat}}(\aleph_n)$ (for $2 \leqslant n \leqslant \omega$) is a theorem of ZFC. Usually, the interest lies in the negations of these hypotheses, as, e. g., Chang's Conjecture at κ, with $\kappa = \varrho^+$, implies $\neg\,\mathrm{TH}(\kappa)$. Correspondingly, $\neg\,\mathrm{TH}^{\mathrm{stat}}(\kappa)$ is a consequence of $\langle\kappa^+,\kappa\rangle \underset{\text{club}}{\twoheadrightarrow} \langle\kappa,<\kappa\rangle$, cf. Lemma 4.2 below. The previous chapter, in showing the consistency of $\langle\omega_2,\omega_1\rangle \underset{\text{club}}{\twoheadrightarrow} \langle\omega_1,<\omega_1\rangle$ from the existence of an ω_1-Erdős cardinal, hence serves to make the consistency of $\neg\,\mathrm{TH}^{\mathrm{stat}}(\kappa)$ plausible for $\kappa > \omega_1$ (although surely larger cardinals are needed than just Erdős-cardinals). In Theorem 4.3, we give a lower bound for the consistency strength of this hypothesis.

4.2 Lemma *Let $\kappa = \varrho^+$ be a cardinal, and assume that $\langle\kappa^+,\kappa\rangle \underset{club}{\twoheadrightarrow} \langle\kappa,<\kappa\rangle$. Then $\neg\,\mathrm{TH}^{\mathrm{stat}}(\kappa)$ holds.*

Proof Assume $\langle\kappa^+,\kappa\rangle \underset{\text{club}}{\twoheadrightarrow} \langle\kappa,<\kappa\rangle$. Assume S, $\langle g_\xi \mid \xi < \kappa^+\rangle$, $\langle D_\xi \mid \xi < \kappa\rangle$ and $\langle D_{\xi\zeta} \mid \xi < \zeta < \kappa\rangle$ satisfied the conditions of $\mathrm{TH}^{\mathrm{stat}}(\kappa)$. Code the sequences by some predicates $G \subseteq \kappa^+ \times \kappa \times \kappa$, and D^1, $D^2 \subseteq \kappa^+ \times \kappa^+ \times \kappa$. Set $\mathfrak{A} := \langle\kappa^+,\kappa,G,D^1,D^2\rangle$. Now apply $\langle\kappa^+,\kappa\rangle \underset{\text{club}}{\twoheadrightarrow} \langle\kappa,<\kappa\rangle$ to find a set $C \subseteq \kappa$ such that C is ϱ-club, and such that for all $\gamma \in C$ there exists some $\mathfrak{B}^\gamma \prec \mathfrak{A}$, $\mathrm{card}(B^\gamma) = \kappa$, $B^\gamma \cap \kappa = \gamma$. As S is stationary and $S \subseteq \mathrm{CF}_\varrho$, there exists a $\gamma \in C \cap S$.

We claim that for any $\xi < \zeta$, both from $B \cap \kappa^+$, γ is an element of $D_{\xi\zeta}$: $D_{\xi\zeta}$ is club in κ, thus as $\mathfrak{B}^\gamma \prec \mathfrak{A}$ and $B^\gamma \cap \kappa = \gamma$, it follows that $D_{\xi\zeta} \cap \gamma$ is club in γ. But then $\sup(D_{\xi\zeta} \cap \gamma) = \gamma \in D_{\xi\zeta}$, as $D_{\xi\zeta}$ is closed.

Now if γ is both in $D_{\xi\zeta}$ and in S, then $g_\xi(\gamma) \neq g_\zeta(\gamma)$ (provided of course that ξ, ζ are both from $B \cap \kappa^+$). Thus $x := \{g_\xi(\gamma) \mid \xi \in B\}$ has cardinality κ. But all the g_ξ are regressive, so that $x \subseteq \gamma < \kappa$, contradiction! Hence $\mathrm{TH}^{\mathrm{stat}}(\kappa)$ must fail. ■

4.3 Theorem *Let κ be a successor cardinal, $\kappa = \varrho^+ \geqslant \omega_2$, $2^\kappa = \kappa^+$. Assume that $\neg\,\mathrm{TH}^{\mathrm{stat}}(\kappa)$. Then 0^{long} exists.*

Proof The proof of this theorem adapts techniques from [Koe88] and [DK83], especially the proof of Theorem B there, as well as those of Ketonen's proof that the existence of an irregular ultrafilter entails the existence of $0^{\#}$ (cf. also Chapter 7).

Let $\kappa = \varrho^+ \geqslant \omega_2$, and assume $\neg\,\mathrm{TH}^{\mathrm{stat}}(\kappa)$. We have to show that 0^{long} exists. So assume to the contrary that 0^{long} does not exist and work for a contradiction.

Let $\mathcal{U}_{\mathrm{can}}$ be the canonical, maximal strong sequence of filters from Definition 2.14. Let $\mathcal{U} := \mathcal{U}_{\mathrm{can}} \upharpoonright \kappa^+$, $\nu := \sup\mathrm{dom}(\mathcal{U})$. Then by Lemma 2.12, $\nu < \kappa^+$.

For $\tau \in (\kappa, \kappa^+)$ let f_τ be a surjection from κ onto τ, and let $F := \{\langle \xi, \zeta, \tau\rangle \mid f_\tau(\xi) = \zeta\}$. Let $F \upharpoonright \tau := F \cap (\kappa \times \tau \times \tau)$.

For $\tau \in (\kappa, \kappa^+)$ such that $\mathrm{cf}(\tau) = \kappa$, let $g_\tau : \kappa \to \tau$ monotone cofinally. If $\mathrm{cf}(\tau) < \kappa$, let $g_\tau : \kappa \to \tau$ be arbitrary. Let $G := \{\langle \xi, \zeta, \tau\rangle \mid g_\tau(\xi) = \zeta\}$. Let $G \upharpoonright \tau := G \cap (\kappa \times \tau \times \tau)$.

Let $h : \kappa^+ \to \mathrm{H}_{\kappa^+}$ be a bijection. Let $\mathcal{H} = \langle \mathrm{H}_{\kappa^+}, \in, h, \mathcal{U}, F, G\rangle$, $W := K_{\kappa^+}[\mathcal{U}]$. For $\tau \in (\kappa, \kappa^+)$, let $\widetilde{\mathcal{H}}_\tau := \mathcal{H} \upharpoonright (h''\tau)$, $H_\tau := h''\tau = |\widetilde{\mathcal{H}}_\tau|$, $W_\tau := H_\tau \cap W$. Let

$$\begin{aligned} E := \{\tau \in (\kappa, \kappa^+) \mid \widetilde{\mathcal{H}}_\tau \prec \mathcal{H} \ \wedge\ \mathrm{On} \cap H_\tau = \tau \\ \wedge\ H_\tau \text{ transitive } \wedge\ \widetilde{\mathcal{H}}_\tau \models W_\tau = K[\mathcal{U}]\} \end{aligned}$$

Then E is a club subset of κ^+.

For $\tau \in E$ let

$$\begin{aligned} K_\tau[\mathcal{U}] &:= W_\tau \\ Q_\tau &:= \langle K_\tau[\mathcal{U}], \in, \mathcal{U} \upharpoonright \tau, F \upharpoonright \tau, f_\tau, G \upharpoonright \tau\rangle \\ \tilde{Q}^\tau_\alpha &:= \text{the smallest } Q \prec Q_\tau \text{ s. t. } \alpha \subseteq Q \\ C_\tau &:= \{\alpha < \kappa \mid \tilde{Q}^\tau_\alpha \cap \kappa = \alpha\} \end{aligned}$$

If x is an element of $K_\tau[\mathcal{U}]$, then $\{\alpha \in C_\tau \mid x \in \tilde{Q}^\tau_\alpha\}$ is a final segment of C_τ. Assume w.l.o.g. that $\min(C_\tau) > \varrho$, where $\varrho^+ = \kappa$. Then for $\alpha \in C_\tau$, $\mathrm{card}(\alpha)^+ = \kappa$. Also, by Lemma 2.12, $\gamma := \sup\mathrm{dom}(\mathcal{U} \upharpoonright \kappa) < \kappa$, so that we may also assume w.l.o.g. that $\min(C_\tau) > \gamma$. It is straightforward to show that C_τ is club in κ.

Let $Q^\tau_\alpha := \langle K^\tau_\alpha, \in, \mathcal{U}^\tau_\alpha, F^\tau_\alpha, (f_\tau)^\tau_\alpha, G^\tau_\alpha \rangle$ be the transitive model isomorphic to $\tilde{Q}^\tau_\alpha$ and let $\tilde{\pi}^\tau_\alpha : Q^\tau_\alpha \to \tilde{Q}^\tau_\alpha$ be the inverse of the collapsing isomorphism. Let $K^\tau_\alpha := (K[\mathcal{U}^\tau_\alpha])^{Q^\tau_\alpha}$, $\tau_\alpha := \mathrm{On} \cap K^\tau_\alpha$, and $\pi^\tau_\alpha := \tilde{\pi}^\tau_\alpha \restriction K^\tau_\alpha$. Then $\pi^\tau_\alpha : K^\tau_\alpha \to K_\tau[\mathcal{U}]$ is an elementary embedding. We refer to this setup as a *Ketonen diagramme*, cf. Figure 4.1.

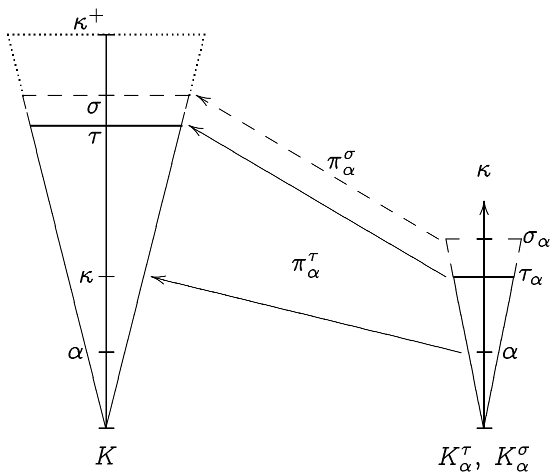

Figure 4.1: The basic structure of the Ketonen diagramme.

Let $\tau, \sigma \in E$, $\tau < \sigma$, $\tau \in \mathrm{rge}(\pi^\sigma_\alpha)$, and $\alpha \in C_\sigma$. Then $\alpha \in C_\tau$ and $Q^\tau_\alpha = (\tilde{\pi}^\sigma_\alpha)^{-1}(Q_\tau) \in Q^\sigma_\alpha$, $\tilde{\pi}^\tau_\alpha = \tilde{\pi}^\sigma_\alpha \restriction Q^\tau_\alpha$, and hence $K^\tau_\alpha = (\pi^\tau_\alpha)^{-1}(K_\tau[\mathcal{U}]) \in K^\sigma_\alpha$, $\pi^\tau_\alpha = \pi^\sigma_\alpha \restriction K^\tau_\alpha$, and $K^\tau_\alpha \prec K^\sigma_\alpha$.

The following lemma, adapted from [DJK81, Lemma 2.3], will play a key rôle in the proof:

4.4 Lemma *Let $S \subseteq \mathrm{CF}_\varrho$ be stationary, $f \in \prod_{\alpha \in S} \mathrm{card}(\alpha)^+$. Let $\tau \in E$. Then there is a $\sigma \in E$, $\sigma > \tau$, such that*

$$\{\alpha \in S \cap C_\sigma \mid f(\alpha) < \sigma_\alpha\} \text{ is stationary.}$$

Proof Assume the statement to be false. Then for all $\sigma \in E$, $\sigma > \tau$, there exists a set $D'_\sigma \subseteq \kappa$ such that D'_σ is club in κ and such that $D'_\sigma \cap \{\alpha \in S \cap C_\sigma \mid f(\alpha) < \sigma_\alpha\} = \emptyset$. Set $D_\sigma := D'_\sigma \cap C_\sigma$. Then D_σ is also club in κ, and for all $\alpha \in D_\sigma \cap S$, $f(\alpha) \geqslant \sigma_\alpha$. For each $\alpha \in S$, let h^α be an injection from $f(\alpha)+1$ into $\mathrm{card}(\alpha)$. For $\sigma \in E$, $\sigma > \tau$, define a function g_σ with $\mathrm{dom}(g_\sigma) = D_\sigma \cap S$ by setting $g_\sigma(\alpha) := h^\alpha(\sigma_\alpha)$. Surely, g_σ is regressive. For $\sigma < \sigma'$ set $D_{\sigma\sigma'} := D_\sigma \cap D_{\sigma'} \cap C_{\sigma\sigma'}$. Then if $\alpha \in D_{\sigma\sigma'} \cap S$, $\sigma_\alpha \neq \sigma'_\alpha$ and hence $g_\sigma(\alpha) \neq g_{\sigma'}(\alpha)$, i. e., $\langle g_\sigma \mid \sigma \in E \setminus (\tau+1)\rangle$ is a $\mathrm{TH}^{\mathrm{stat}}(\kappa)$-sequence, contradiction. $\square$(Lemma 4.4)

We now start to distinguish several cases. One of two things will happen: Either, as in Case 1, we can define, from one of the embeddings π^τ_α, an ultrafilter on some α which allows us to get an embedding from $K[\mathcal{U}]$ to $K[\mathcal{U}]$, contradicting the rigidity of $K[\mathcal{U}]$. Or we can, as in Case 2.1, show that some ultrafilter has been omitted in the definition of the canonical sequence, equally a contradiction.

Case 1 $\mathrm{dom}(\mathcal{U}) \cap [\kappa, \kappa^+) = \emptyset$.

In this case, there is no ultrafilter from the canonical sequence in the interval from κ to κ^+. By Lemma 2.12 we now know that $\nu = \sup\mathrm{dom}(\mathcal{U}) = \sup\mathrm{dom}(\mathcal{U} \restriction \kappa) < \kappa$. By restricting to the right sets of τ's and α's, we can mimick the original Ketonen proof, as the K^τ_α condensate more or less to initial segments of $K[\mathcal{U}]$.

For the rest of this case, assume that for all $\tau \in E$, $\min(C_\tau) > \nu$. Also, as $d := \mathrm{dom}(\mathcal{U}) \subseteq \nu < \kappa < \kappa^+$, $d \in K[\mathcal{U}]$, we have that $d \in \tilde{Q}^\tau_\alpha$ for a final segment of C_τ, so again assume w.l.o.g. that this holds for all $\alpha \in C_\tau$. Then, as $\nu < \alpha$, we have that $\mathrm{dom}(\mathcal{U}^\tau_\alpha) = \mathrm{dom}(\mathcal{U})$. Similarly argue that $\mathcal{U} \in \tilde{Q}^\tau_\alpha$, so $\mathcal{U}^\tau_\alpha = (\pi^\tau_\alpha)^{-1}(\mathcal{U})$, and if $\sigma \in E$, $\tau < \sigma$, and $\tau \in \mathrm{rge}(\pi^\sigma_\alpha)$, then $\mathcal{U}^\tau_\alpha = \mathcal{U}^\sigma_\alpha$.

The following assertion $(\star)_\kappa$ is a variation of the property $(*)_\lambda$ from the proof of [DK83, Theorem B].

$(\star)_\kappa$ Let $\tau \in E$, $S \subseteq C_\tau \cap \mathrm{CF}_\varrho$, S stationary, and let $\langle M_\alpha \mid \alpha \in S\rangle$ be a sequence of short, iterable $\mathcal{U}^\tau_\alpha$-premice, $\mathrm{card}(M_\alpha) < \mathrm{card}(\alpha)^+ = \kappa$. Then there is $\sigma \in E$, $\sigma > \tau$, and S' stationary, $S' \subseteq S \cap C_\sigma$, such that

$$\forall \alpha \in S' \left(\tau \in \mathrm{rge}(\pi^\sigma_\alpha) \ \wedge\ \mathrm{lp}(M_\alpha) \subseteq K^\sigma_\alpha\right).$$

Claim 1 $(\star)_\kappa$ holds.

Proof By Theorem 2.16 we have that for limit ordinals $\eta \in (\max(\nu,\kappa),\kappa^+)$ with $\mathrm{cf}(\eta) < \mathrm{card}(\eta)$

$$K[\mathcal{U}_{\mathrm{can}}] \models \text{“}\eta \text{ is singular”}.$$

Thus

$$K_{\kappa^+}[\mathcal{U}] \models \text{“}\eta \text{ is singular”}.$$

By our choice of G we then get that

$$\forall \eta \in (\max(\nu,\kappa),\kappa^+)\ \ \langle K_{\kappa^+}[\mathcal{U}], F, G\rangle \models \text{“}g_\eta \text{ does not map } \kappa \text{ monotone cofinally into } \eta \implies \eta \text{ is singular”}.$$

Let τ, S, $\langle M_\alpha \mid \alpha \in S\rangle$ be given, and let $\mu_\alpha := \min\mathrm{meas}(M_\alpha)$. By iterating M_α with its least measure if necessary, we can assume w.l.o.g. that $\mu_\alpha > \alpha$ and thus $\mu_\alpha > \nu$, while still keeping $\mu_\alpha < \kappa$. Then the ω_1-st iterate $\mu_\alpha^{(\omega_1)}$ of μ_α is also less than $\kappa = \mathrm{card}(\alpha)^+$. Thus by Lemma 4.4 there is a $\sigma \in E$, $\sigma > \tau$, and S' stationary, $S' \subseteq S \cap C_\sigma$, such that for $\alpha \in S'$ we have that $\tau \in \mathrm{rge}(\pi^\sigma_\alpha)$ and $\mu_\alpha^{(\omega_1)} < \sigma_\alpha$. Now set $\mu^*_\alpha := \mu_\alpha^{(\omega)}$ or $:= \mu_\alpha^{(\omega_1)}$ and $M^*_\alpha := M_\alpha^{(\omega)}$ or $:= M_\alpha^{(\omega_1)}$, ensuring that $\mathrm{cf}(\mu^*_\alpha) \neq \mathrm{cf}(\alpha)$. Then, setting $\eta := \pi^\sigma_\alpha(\mu^*_\alpha)$,

$$\langle K^\sigma_\alpha, F^\sigma_\alpha, G^\sigma_\alpha\rangle \models \text{“}(g^\sigma_\alpha)_{\mu^*_\alpha} \text{ does not map } \alpha \text{ monotone cofinally into } \mu^*_\alpha\text{”},$$

so

$$\langle K_{\kappa^+}[\mathcal{U}], F, G\rangle \models \text{“}g_\eta \text{ does not map } \kappa \text{ monotone cofinally into } \eta\text{”}.$$

As $\eta \in (\max(\nu,\kappa),\kappa^+)$, it follows that

$$\langle K_{\kappa^+}[\mathcal{U}], F, G\rangle \models \text{“}\eta \text{ is singular”},$$

and so

$$\langle K^\sigma_\alpha, F^\sigma_\alpha, G^\sigma_\alpha\rangle \models \text{“}\mu^*_\alpha \text{ is singular”}.$$

As $\mathcal{U}^\sigma_\alpha = \mathcal{U}^\tau_\alpha$, Theorem 2.17 now implies that $\mathrm{lp}(M_\alpha) \subseteq K^\sigma_\alpha$. □(Claim 1)

Now pick some $\tau \in E$.

Claim 2 The set $S_1 := \{\alpha \in C_\tau \cap \mathrm{CF}_\varrho \mid \mathcal{U}^\tau_\alpha$ is strong$\}$ is stationary.

Proof Assume not. Then there is some closed unbounded set D such that $D \cap S_1 = \emptyset$, and for any $\alpha \in D \cap C_\tau \cap \mathrm{CF}_\varrho$, $K[\mathcal{U}^\tau_\alpha] \models$ "$\mathcal{U}^\tau_\alpha$ is not a sequence of measures". This is witnessed, for some $\zeta \in \mathrm{dom}(\mathcal{U}^\tau_\alpha)$, by some $x \in \mathcal{P}(\zeta) \cap K[\mathcal{U}^\tau_\alpha]$, or $x \in {}^\zeta\zeta \cap K[\mathcal{U}^\tau_\alpha]$, or $x \in {}^\xi\mathcal{P}(\zeta) \cap K[\mathcal{U}^\tau_\alpha]$, for some $\xi < \zeta$. $x_\alpha := x$ is either a set not measured by $\mathcal{U}^\tau_\alpha(\zeta)$, or a non-constant regressive function, or a sequence of sets of the filter whose intersection is not in the filter. In any case, $x \in K_{\zeta^{+K[\mathcal{U}^\tau_\alpha]}}[\mathcal{U}^\tau_\alpha]$, and $\zeta^{+K[\mathcal{U}^\tau_\alpha]} \leqslant \kappa$. Thus, for $\alpha \in D \cap C_\tau \cap \mathrm{CF}_\varrho$, we can find a short iterable $\mathcal{U}^\tau_\alpha$-premouse M_α, $\mathrm{card}(M_\alpha) < \kappa$, such that $x_\alpha \in \mathrm{lp}(M_\alpha)$. $(\star)_\kappa$ implies that there is a $\sigma \in E$, $\sigma > \tau$, and S' stationary, $S' \subseteq D \cap C_\tau \cap \mathrm{CF}_\varrho$, such that for all $\alpha \in S'$, $\tau \in \mathrm{rge}(\pi^\sigma_\alpha)$ and $x_\alpha \in \mathrm{lp}(M_\alpha) \subseteq K^\sigma_\alpha$. As $\mathcal{U}^\sigma_\alpha = \mathcal{U}^\tau_\alpha$, $K^\sigma_\alpha \models$ " $\mathcal{U}^\sigma_\alpha$ is not a sequence of measures". But this is a contradiction.

$\square$(Claim 2)

For $\alpha \in S_1$, $\mathcal{U}^\tau_\alpha$ is strong and $\mathrm{dom}(\mathcal{U}^\tau_\alpha) = \mathrm{dom}(\mathcal{U})$, so that Theorem 2.13 implies that $\mathcal{U}^\tau_\alpha = \mathcal{U}$. This is what we meant when we said that the K^τ_α condensate more or less to initial segments of $K[\mathcal{U}]$.

Claim 3 κ is inaccessible in $K[\mathcal{U}]$.

Proof Assume to the contrary that $\kappa = \lambda^{+K[\mathcal{U}]}$. We can assume that $\min(C_\tau) > \lambda$. Thus for each $\alpha \in S_1$, $K[\mathcal{U}] \models$ "α is not a cardinal". Pick for each such α a short iterable $\mathcal{U}$-premouse M_α, $\mathrm{card}(M_\alpha) < \kappa$, such that there is a surjection $h_\alpha : \lambda \to \alpha$, $h_\alpha \in \mathrm{lp}(M_\alpha)$. By Lemma 4.4 there is a $\sigma \in E$, $\sigma > \tau$, and S stationary, $S' \subseteq S_1 \cap C_\sigma$, such that for all $\alpha \in S'$, $\tau \in \mathrm{rge}(\pi^\sigma_\alpha)$ and $\mathrm{lp}(M_\alpha) \subseteq K^\sigma_\alpha$. But then $K^\sigma_\alpha \models$ "α is not a cardinal", clearly a contradiction, as $\pi^\sigma_\alpha(\alpha) = \kappa$. $\square$(Claim 3)

Claim 4 The set $S_2 := \{\alpha \in S_1 \mid \mathcal{P}(\alpha) \cap K[\mathcal{U}] \in K^\tau_\alpha\}$ is stationary.

Proof By the last claim, $\mathcal{P}(\alpha) \cap K[\mathcal{U}] \in K_\kappa[\mathcal{U}]$ for $\alpha < \kappa$. Pick a sequence $\langle M_\alpha \mid \alpha \in S_1 \rangle$ of short iterable $\mathcal{U}$-premice such that $\mathcal{P}(\alpha) \cap K[\mathcal{U}] \in \mathrm{lp}(M_\alpha)$. Then, again, there is a $\sigma \in E$, $\sigma > \tau$, and S' stationary, $S' \subseteq S_1 \cap C_\sigma$, such that for $\alpha \in S'$, $\tau \in \mathrm{rge}(\pi^\sigma_\alpha)$ and $\mathcal{P}(\alpha) \cap K[\mathcal{U}] \in K^\sigma_\alpha$. As $K^\sigma_\alpha \subseteq K[\mathcal{U}]$, we get that $\mathcal{P}(\alpha) \cap K[\mathcal{U}] = \mathcal{P}(\alpha)^{K^\sigma_\alpha}$. Since $K^\tau_\alpha \prec K^\sigma_\alpha$, $K^\tau_\alpha \models$ "$\mathcal{P}(\alpha)$ exists", and so

$\mathcal{P}(\alpha)^{K^\tau_\alpha} = \mathcal{P}(\alpha)^{K^\sigma_\alpha} = \mathcal{P}(\alpha) \cap K[\mathcal{U}] \in K^\tau_\alpha$. S_2 is obviously a superset of S' and hence is also stationary. $\square$(Claim 4)

For $\alpha \in S$ define $K[\mathcal{U}]$-ultrafilters $\mathcal{V}^\tau_\alpha$ by

$$x \in \mathcal{V}^\tau_\alpha :\leftrightarrow \alpha \in \pi^\tau_\alpha(x).$$

Claim 5 There is an $\alpha \in S_2$ such that $\mathrm{Ult}(K_\kappa[\mathcal{U}], \mathcal{V}^\tau_\alpha)$ is well-founded.

Proof Assume not. Then for each $\alpha \in S_2$ there are functions $f^\alpha_0, f^\alpha_1, \ldots \in K_\kappa[\mathcal{U}]$ such that for all $i \in \omega$, $\{\xi < \alpha \mid f^\alpha_{i+1}(\xi) \in f^\alpha_i(\xi)\} \in \mathcal{V}^\tau_\alpha$. Pick short iterable $\mathcal{U}$-premice M_α, $\mathrm{card}(M_\alpha) < \kappa$, such that $f^\alpha_i \in \mathrm{lp}(M_\alpha)$. Find $\sigma \in E$, $\sigma > \tau$, and S' stationary, $S' \subseteq S_2 \cap C_\sigma$, such that for all $\alpha \in S'$, $\tau \in \mathrm{rge}(\pi^\sigma_\alpha)$ and $\mathrm{lp}(M_\alpha) \subseteq K^\sigma_\alpha$. Then

$$\begin{aligned} \alpha \in \pi^\tau_\alpha &\left(\{\xi < \alpha \mid f^\alpha_{i+1}(\xi) \in f^\alpha_i(\xi)\}\right) \\ &= \pi^\sigma_\alpha \left(\{\xi < \alpha \mid f^\alpha_{i+1}(\xi) \in f^\alpha_i(\xi)\}\right) \\ &= \{\xi < \kappa \mid \pi^\sigma_\alpha(f^\alpha_{i+1})(\xi) \in \pi^\sigma_\alpha(f^\alpha_i)(\xi)\}. \end{aligned}$$

This gives an infinite descending sequence of ordinals,

$$\ldots \in \pi^\sigma_\alpha(f^\alpha_{i+1})(\alpha) \in \pi^\sigma_\alpha(f^\alpha_i)(\alpha) \in \ldots \in \pi^\sigma_\alpha(f^\alpha_0)(\alpha).$$

$\square$(Claim 5)

Let $\alpha \in S_2$ be such that $\mathrm{Ult}(K_\kappa[\mathcal{U}], \mathcal{V}^\tau_\alpha)$ is well-founded, and let π denote the canonical embedding of this ultrapower. We thus have $\kappa \geqslant \omega_1$, $\kappa > \sup \mathrm{dom}(\mathcal{U})$, and $\pi : K_\kappa[\mathcal{U}] \to W$, $\mathrm{crit}(\pi) = \alpha > \nu$. Theorem 2.18 implies that there is an elementary embedding $\tilde{\pi} : K[\mathcal{U}] \to K[\mathcal{U}]$ with the same critical point. As κ is inaccessible in $K[\mathcal{U}]$ and $\kappa \geqslant \omega_2$, Theorem 2.15 implies that there is $\mathcal{U}' >_e \mathcal{U}$ with $\eta := \min \mathrm{dom}(\mathcal{U}' \setminus \mathcal{U})$ satisfying $\eta \geqslant \alpha > \nu$ and $\eta < \kappa$. But this contradicts the definition of $\mathcal{U} = \mathcal{U}_{\mathsf{can}} \restriction \kappa$. $\square$(Case 1)

Case 2 $\mathrm{dom}(\mathcal{U}) \cap [\kappa, \kappa^+) \neq \emptyset$.

Again let $\nu := \sup \mathrm{dom}(\mathcal{U})$.

Case 2.1 $\kappa^+ > \nu^{+K[\mathcal{U}]}$.

We have $\mathcal{U} \in K_{\kappa^+}[\mathcal{U}]$. W.l.o.g. assume that for all $\tau \in E$, $\mathcal{U} \in K_\tau[\mathcal{U}]$. Furthermore, assume that for all $\alpha \in C_\tau$, $\mathcal{U} \in \tilde{Q}^\tau_\alpha$. Then $\mathcal{U}^\tau_\alpha = (\pi^\tau_\alpha)^{-1}(\mathcal{U})$, and if $\tau \in \mathrm{rge}(\pi^\sigma_\alpha)$, then $\mathcal{U}^\sigma_\alpha = \mathcal{U}^\tau_\alpha$. Pick some $\tau \in E$.

Claim 6 $(\star)_\kappa$ holds.

Proof Assume w.l.o.g. that for $\alpha \in C_\tau$, $\nu \in \tilde{Q}^\tau_\alpha$, and let $\nu^\tau_\alpha := (\pi^\tau_\alpha)^{-1}(\nu)$. Note that $\nu \geqslant \kappa$. Then as before we have

$$\forall \eta \in [\nu+1, \kappa^+) \;\; \langle K_{\kappa^+}[\mathcal{U}], F, G\rangle \vDash \text{“}g_\eta \text{ does not map } \kappa \text{ monotone cofinally into } \eta \implies \eta \text{ is singular”}.$$

Given the sequence $\langle M_\alpha \mid \alpha \in S\rangle$, proceed as before. Iterate M_α initially with its least measure to ensure not only $\mu_\alpha > \alpha$ but also $\mu_\alpha > \nu^\tau_\alpha$. Then $\eta := \pi^\sigma_\alpha(\mu^*_\alpha)$ will be larger than $\pi^\sigma_\alpha(\nu^\tau_\alpha) = \nu$, so we can continue as in Case 1. □(Claim 6)

Claim 7 The set $S_1 := \{\alpha \in C_\tau \cap \mathrm{CF}_\varrho \mid \mathcal{U}^\tau_\alpha \text{ is strong}\}$ is stationary.

Proof As before. □(Claim 7)

Now pick an α from S_1. $\mathcal{U}^\tau_\alpha$ is strong, so $\mathcal{U}^\tau_\alpha \restriction \alpha$ is strong, too. Furthermore, $\mathrm{dom}(\mathcal{U}^\tau_\alpha \restriction \alpha) = \mathrm{dom}(\mathcal{U} \restriction \alpha) = \mathrm{dom}(\mathcal{U} \restriction \kappa)$, as we assured $\alpha > \gamma = \sup \mathrm{dom}(\mathcal{U} \restriction \kappa)$ at the outset. Thus by Theorem 2.13, $\mathcal{U}^\tau_\alpha \restriction \alpha = \mathcal{U} \restriction \kappa$. Hence $\mathcal{U}^\tau_\alpha >_e \mathcal{U} \restriction \kappa$. Now $\min \mathrm{dom}(\mathcal{U}^\tau_\alpha \backslash \mathcal{U} \restriction \kappa) = \min \mathrm{dom}(\mathcal{U}^\tau_\alpha \backslash \mathcal{U}^\tau_\alpha \restriction \alpha) < \kappa$, as $\mathrm{dom}(\mathcal{U}) \cap [\kappa, \kappa^+) \neq \emptyset$. This contradicts the definition of $\mathcal{U} = \mathcal{U}_{\mathrm{can}} \restriction \kappa^+$ (cf. Figure 4.2). □(Case 2.1)

Case 2.2 $\kappa^+ = \nu^{+K[\mathcal{U}]}$.

Case 2.2.1 $\nu \notin \mathrm{dom}(\mathcal{U})$.

This is similar to Case 2.1. We have $d := \mathrm{dom}(\mathcal{U}) \subseteq \nu$, so $d \in K_{\kappa^+}[\mathcal{U}]$, hence w.l.o.g. $d \in \tilde{Q}^\tau_\alpha$ for all relevant τ and α. Set $d^\tau_\alpha := (\pi^\tau_\alpha)^{-1}(d)$, so $d^\tau_\alpha = \mathrm{dom}(\mathcal{U}^\tau_\alpha)$, and if $\tau \in \mathrm{rge}(\pi^\sigma_\alpha)$, then $d^\sigma_\alpha = d^\tau_\alpha$. But then $\mathcal{U}^\sigma_\alpha = \mathcal{U}^\tau_\alpha$, too. For if $\xi \in d^\tau_\alpha$, then $\zeta := \pi^\tau_\alpha(\xi) = \pi^\sigma_\alpha(\xi) \in d$, and $\mathcal{U}^\tau_\alpha(\xi) = (\pi^\tau_\alpha)^{-1}(\mathcal{U}(\zeta)) = (\pi^\sigma_\alpha)^{-1}(\mathcal{U}(\zeta)) = \mathcal{U}^\sigma_\alpha(\xi)$. Note that $\mathcal{U}(\zeta) \in \tilde{Q}^\tau_\alpha$ as it is definable from ζ and U.

The rest of the proof is as Case 2.1. Note that we need neither the functions g_η nor Theorem 2.16 to show that $(\star)_\kappa$ holds, as $K_{\kappa^+}[\mathcal{U}] \vDash$ “η is singular” for any $\eta \in (\nu, \kappa^+)$, anyway. □(Case 2.2)

Case 2.2.2 $\nu \in \mathrm{dom}(\mathcal{U})$.

In this case, there is an ultrafilter “at ν”. To proceed, we will just cut this ultrafilter away from the sequence and try to continue as before: If there is

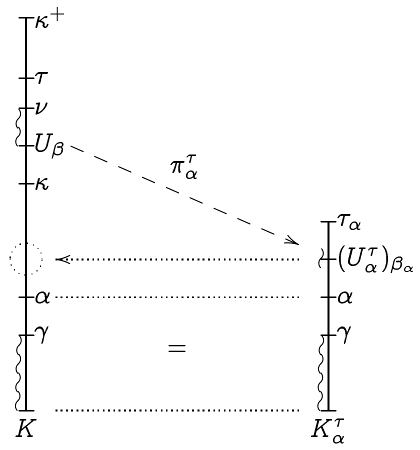

Figure 4.2: The condensated substructure K^τ_α provides an ultrafilter $(U^\tau_\alpha)_{\beta_\alpha}$ that should have been included in the canonical sequence, as indicated by the dotted circle.

some ultrafilter left, at some $\zeta \in [\kappa, \nu)$, we copy Case 2.1, which will happen in Case 2.2.2.1. Otherwise, in Case 2.2.2.2, there is no ultrafilter with critical point between κ and ν, which makes it possible to apply the method of Case 1.

Case 2.2.2.1 $\exists \zeta \in \mathrm{dom}(\mathcal{U}) \cap [\kappa, \nu)$.

Set $\mathcal{U}' := \mathcal{U} \restriction \nu$. Note that as in Case 2.2.1 $K_{\kappa^+}[\mathcal{U}] \models$ "η is singular" for any $\eta \in (\nu, \kappa^+)$. By Theorem 2.10, $\mathcal{P}(\nu) \cap K[\mathcal{U}] = \mathcal{P}(\nu) \cap K[\mathcal{U} \restriction \nu]$, so $K_{\kappa^+}[\mathcal{U} \restriction \nu] \models$ "η is singular", too. Now consider the "$'$-version" of the Ketonen-diagramme, i. e., define $Q'_\tau, E', \widetilde{Q}'^\tau_\alpha, \mathcal{U}'^\tau_\alpha$, etc., starting out from $K[\mathcal{U}']$ instead of $K[\mathcal{U}]$. Then show that the corresponding property $(\star')_\kappa$ holds, noting that by the above remark $K[\mathcal{U}'] \models$ "η is singular". Continue as in Case 2.1 to see that $\mathcal{U}'^\tau_\alpha$ is strong for sufficiently many α and then derive a contradiction as before. $\Box$(Case 2.2.2.1)

Case 2.2.2.2 $\mathrm{dom}(\mathcal{U}) \cap [\kappa, \nu] = \{\nu\}$.

The proof of this case is a combination of the proofs of Case 2.2.2.1 and Case 1. Again, build the "$'$-version" of the Ketonen-diagramme and show that $(\star')_\kappa$ holds. Prove that $\mathcal{U}'^\tau_\alpha$ is strong for sufficiently many α. We cannot, as in

Case 2.2.2.1, derive a contradiction from this just yet, as there is no ultrafilter on the ${\mathcal{U}'}^{\tau}_{\alpha}$-sequence above α. But we can show, returning to Case 1, that if ${\mathcal{U}'}^{\tau}_{\alpha}$ is strong and $\alpha > \nu' := \sup \operatorname{dom}(\mathcal{U}')$ ($\nu' < \kappa$ by our assumption), then ${\mathcal{U}'}^{\tau}_{\alpha} = \mathcal{U}'$, as $\operatorname{dom}({\mathcal{U}'}^{\tau}_{\alpha}) = \operatorname{dom}(\mathcal{U}')$. Thus we can continue as in Case 1 to prove that κ is inaccessible in $K[\mathcal{U}']$. The definition of the $K[\mathcal{U}']$-ultrafilters $\mathcal{V}^{\tau}_{\alpha}$ also goes through. Note that $\sup \operatorname{dom}(\mathcal{U}') < \kappa$. Thus as before we get an elementary embedding $\pi : K_{\kappa}[\mathcal{U}'] \to W$ with $\operatorname{crit}(\pi) = \alpha > \nu'$. This leads to the same contradiction as in Case 1. $\square$(Case 2.2.2.2)

This completes the proof of the theorem. $\blacksquare$

$\neg$ TH implies the *weak Chang's Conjecture* (cf. [DK83]). One could try to extend the above proof to start from the assumption of wCC(κ) instead of $\neg\,\mathrm{TH}^{\mathsf{stat}}(\kappa)$. However, this fails for the same reason as the attempt to start from just $\neg$ TH instead of $\neg\,\mathrm{TH}^{\mathsf{stat}}$: the proof requires a repeated application of the assumption (or rather of Lemma 4.4). But $\neg$ TH needs as an "input" at least a club set, yielding only a stationary set as "output"; wCC, in fact, gives even just one single α. In contrast, although $\neg\,\mathrm{TH}^{\mathsf{stat}}$ also just yields a stationary set, that is also enough as "input" for a second application (or, indeed, a finite number of them). In Chapter 7, we will show how infinite applications can yield even stronger results, albeit starting from a stronger premise, the existence of an irregular ultrafilter.

Chapter 5

Core Models up to a Strong Cardinal

"What do you keep that mouse for?" I said. "You should bury it, or throw it into the lake."

"Why, it's to measure with!" cried Bruno.

(Lewis Carroll, Bruno's Revenge)

To cope with inner models for larger large cardinals, in our case anything up to a strong cardinal, we need to extend our concept of mouse. Every year seems to bring a new definition of this fundamental concept of core model theory, encompassing ever larger hypotheses. For our purposes, the finestructure developed in [Koe89] will suffice completely. [Sch96] goes on to show how one can build a decent core model from these mice. As in Chapter 2, we now proceed to review some basic definitions and theorems. The original numbering from [Sch96] is given in parentheses. Any terms not defined here can be found there or in [Koe89].

5.1 Definition A predicate E is *natural* iff

i) $\forall z \in E\ \exists \nu \in \mathrm{Lim}\ \exists \kappa < \nu\ \exists a \in [\nu]^{<\omega}\ \exists x \in \mathcal{P}\left([\kappa]^{\mathrm{card}(a)}\right)\ \left(z = \langle \nu, \kappa, a, x\rangle\right)$.

ii) $\forall \langle \nu, \kappa, a, x\rangle, \langle \nu', \kappa', a', x'\rangle \in E\ \left(\nu = \nu' \ \rightarrow\ \kappa = \kappa'\right)$.

$\kappa(\nu) = \kappa^E(\nu)$ is the unique κ such that, for some a and x, $\langle \nu, \kappa, a, x\rangle$ is in E. For $\nu \in \mathrm{On}$ set $E_\nu := \{\langle a, x\rangle \mid \langle \nu, \kappa(\nu), a, x\rangle \in E\}$, and for $a \in [\nu]^{<\omega}$, set $E_{\nu,a} := \{x \mid \langle a, x\rangle \in E_\nu\}$. Let $\mathrm{dom}(E) := \{\nu \in \mathrm{On} \mid E_\nu \neq \emptyset\}$, and for $X \subseteq \mathrm{On}$ let $E \restriction X := \{\langle \nu, \kappa, a, x\rangle \mid \nu \in X\}$.

5.2 Definition Let E be a natural predicate and $M := \langle \mathrm{J}_\alpha[E], F\rangle$, for some $\alpha \in \mathrm{On}$ (or $\alpha = \mathrm{On}$), be an acceptable model. For $\nu < \alpha$, let $E^M_\nu := E_\nu$ and set $E^M_\alpha := F$. Then M is called an *extender structure* iff

i) $\forall \nu \in \mathrm{dom}(E)\ \big(E_\nu \subseteq \mathrm{J}_\nu[E]$ is a 0-folded extender at $\kappa(\nu), \nu$ on $\mathrm{J}_\nu[E]\big)$,

ii) E is *coherent*, i. e., for any $\nu \in \mathrm{dom}(E)$, if $N := \mathrm{Ult}(\mathrm{J}_\nu[E], E_\nu)$, then $E^N \restriction \nu = E \restriction \nu$ and $E^N_\nu = \emptyset$,

iii) $\forall \nu \in \mathrm{dom}(E)\ \forall \xi \in [\kappa(\nu)^{+\mathrm{J}_\nu[E]}, \nu]\ \big(\xi \in \mathrm{dom}(E)$ and E_ξ is an extender at $\kappa(\nu)$, ξ on $\mathrm{J}_\xi[E]\big)$,

iv) E is *non-overlapping*, i. e., for any $\nu \in \mathrm{dom}(E)$ and any λ less than $\kappa(\nu)$, $\sup\{\xi < \kappa(\nu) \mid \kappa(\xi) < \lambda\} < \kappa(\nu)$.

5.3 Definition Let $\alpha \in \mathrm{On}$ and let $M = \langle J_\alpha[E], E_\alpha\rangle$ be an extender structure. Let $\kappa \in \mathrm{Card}^M$. Then define

i) $O^M(\kappa)$ to be the smallest limit ordinal greater than or equal to κ^{+M} which is a strict upper bound of $\{\xi \leqslant \alpha \mid E_\xi \neq \emptyset\}$,

ii) $o^M(\kappa) := (O^M(\kappa) - \kappa^{+M})/\omega$, the *Mitchell order* of κ in M.

M is called *topless* if $E_\alpha = \emptyset$. For $\beta \leqslant \alpha$ set $M|\beta := \langle \mathrm{J}_\beta[E], E_\beta\rangle$ and $M \downarrow \beta := \langle \mathrm{J}_\beta[E], \emptyset\rangle$. These are both extenderstructures, $M \downarrow \beta$ being the topless version of $M|\beta$.

Let N be another extenderstructure. Then M is an *initial segment* of N and N an *end-extension* of M, denoted by $M \subseteq N$, iff $M = N|(\mathrm{On} \cap M)$ or $M = N\downarrow(\mathrm{On} \cap M)$. If $M \subseteq N$ and $M \neq N$, then M is a *proper initial segment* of N.

5.4 Definition (1.2) An extender structure $M = \langle \mathrm{J}_\alpha[E], E_\alpha\rangle$ is called *weakly amenable* iff $E_\alpha \neq \emptyset$ and if for all sequences $\langle x_\xi \mid \xi < \kappa(\alpha)\rangle$ from $\mathrm{J}_\alpha[E]$ and for any $a \in [\alpha]^{<\omega}$ the set $\{\xi < \kappa(\alpha) \mid x_\xi \in E_{\alpha,a}\}$ is in $\mathrm{J}_\alpha[E]$.

5.5 Lemma (1.2) *Assume $M = \langle \mathrm{J}_\alpha[E], E_\alpha\rangle$ is an extender structure such that $N := \mathrm{Ult}(M, E_\alpha)$ is transitive. Then*

$$M \text{ is weakly amenable } \leftrightarrow \mathcal{P}(\kappa(\alpha)) \cap N = \mathcal{P}(\kappa(\alpha)) \cap M.$$

Premice and *(coarse and fine) ultrapowers* are defined in a straightforward way. An *iteration* is given by the sequence of indices $\langle \nu_i \mid i < \vartheta \rangle$, indicating which extender to use to compute the next ultrapower, the sequence of cutback-points $\langle \alpha_i \mid i < \vartheta \rangle$, and the sequence of indicators $\langle r_i \mid i < \vartheta \rangle$, determining whether coarse or fine ultrapowers should be taken. An iteration is *non-degenerate* if only finitely many α_i are less than $\mathrm{On} \cap M^i$. A premouse M is *iterable above* τ if any iteration with indices at least τ can be freely continued. M is a *mouse* if it is iterable above 0. An iteration is *simple* if no cutbacks occur, i. e., if for all i less than ϑ, α_i is equal to $\mathrm{On} \cap M^i$.

Mice are finely coiterable. If M^* and N^* denote the coiterates of respectively M and N and if $M^* \subseteq N^*$, then we say $M \leqslant^* N$. Also, the iteration of M to M^* is simple in this case.

If M is a mouse above τ, then there exists a unique premouse M_0 such that M_0 is τ-sound and M is a simple fine iterate of M_0 above $\max\{\tau, \varrho^\omega_M\}$. This is the *$\tau$-core* of M, denoted by $\mathrm{core}_\tau(M)$. The *core* of M is its 0-core.

5.6 Definition (1.14) Let $M = \mathrm{J}_\alpha[E]$ be an extender structure. Then let κ^M be defined as follows:

i) If there exists some $\xi \leqslant \alpha$ such that E_ξ is an extender at κ, ξ on M and for all $\zeta \in [\xi, \alpha)$, E_ζ is an extender at κ, ζ on M, then set $\kappa^M := \kappa$,

ii) otherwise, let $\kappa^M := \sup(\mathrm{Card}^M)$.

Thus, if $E_\alpha \neq \emptyset$, then $\kappa^M = \mathrm{crit}(E_\alpha)$.

5.7 Definition (1.15) Let $M = \langle \mathrm{J}_\alpha[E], E_\alpha \rangle$ be a premouse such that $\kappa^M < \alpha$. A *collapsing mouse for M* is a premouse $N = \langle \mathrm{J}_\beta[E'], E'_\beta \rangle$ such that

i) $M \subseteq N$,

ii) $\mathrm{sq}(N)$ is a κ^M-sound mouse above κ^M,

iii) $\mathcal{P}\left(\kappa^M\right) \cap N \subseteq M$, but

iv) $\varrho^\omega_{\mathrm{sq}(N)} \leqslant \kappa^M$.

In this case, $\mathrm{sq}(N)$ is called a *squashed collapsing mouse for M* (for details on squashed and stretched mice, see [Sch96]). N is thus a "minimal" end-extension of M over which $(\kappa^M)^{+M}$ is collapsed onto κ^M. M is *presolid* if $\kappa^M < \alpha$ and there is *no* collapsing mouse for M.

5.8 Lemma (1.16) *Collapsing mice are unique: let M be a premouse such that N and Q are collapsing mice for M. Then $N = Q$.*

5.9 Corollary (1.17) *Let M and N be coarse mice such that for some $\kappa < \mathrm{On} \cap M \cap N$, $M \downarrow \kappa = N \downarrow \kappa$, and $E^M_\kappa = E^N_\kappa = \emptyset$. Let $\nu := \min\{\kappa^{+M}, \kappa^{+N}\}$. Then $M \downarrow \nu = N \downarrow \nu$.*

5.10 Definition (1.19) Let $M = \langle \mathrm{J}_\alpha[E], E_\alpha \rangle$ be a premouse, $\beta \leqslant \alpha$. M is called *neat beyond β* provided

i) $\forall \gamma < \alpha$ ($M|\gamma$ is a mouse),
ii) $\forall \gamma \in [\beta, \alpha]$ ($E^M_\gamma \neq \emptyset \to E^M_\gamma$ is countably complete).

M is *neat* if it is neat beyond 0.

Any neat premouse is a mouse [Sch96, Lemma 1.20].

5.11 Definition (1.21) Let $M = \langle \mathrm{J}_\alpha[E], E_\alpha \rangle$ be a premouse, $\beta \leqslant \alpha$. An iteration $\mathcal{I}$ of M is called *beyond β* if all indices ν used in the iteration are greater than or equal to β. M is called *(coarsely, finely) iterable beyond β* if any non-degenerate (coarse, fine) iteration of M beyond β can be freely continued beyond β.

M is called *prestrong* provided the following holds: Any premouse N end-extending M which is finely iterable beyond $\alpha + \omega$ (or α, if $E^M_\alpha = \emptyset$) is, in fact, a mouse.

M is called *strong* if M is prestrong and for any mouse N end-extending M, M is an initial segment of core(N).

M is called *solid* if M is presolid and prestrong.

Hence every prestrong premouse is a mouse, and every strong mouse is sound. Moreover, neat mice are prestrong [Sch96, Lemma 1.22].

5.12 Definition (1.26) $\mathrm{L}^{\mathrm{strong}}$ denotes the statement that there exists an inner model with a strong cardinal. $\neg\,\mathrm{L}^{\mathrm{strong}}$ denotes the negation of this statement.

5.13 Definition (1.27) A J-model $M = \langle \mathrm{J}_\alpha[E], U \rangle$ is called a *pistol-premouse (p-premouse)* if

i) $\bar{M} := \mathrm{J}_\alpha[E]$ is a topless premouse such that $\kappa := \kappa^{\bar{M}}$ is the largest cardinal in $\bar{M}$.

ii) There is some $\lambda < \kappa$ such that $O^{\bar{M}}(\lambda) \geqslant \kappa$ (in which case one says that κ is *overlapped*[1].

iii) $M \models$ "U is a non-trivial κ-complete normal ultrafilter on κ".

M is called *neat* if $\bar{M}$ is neat above λ and U is countably complete.

5.14 Lemma (1.27) *If there is a neat p-premouse, then* $\mathrm{L}^{\text{strong}}$ *holds.*

Next, we present a result on upward extensions of embeddings. The construction itself is defined in detail in [Sch96, Chapter 4], which also contains the proofs of the following statements. Starting from a premouse $N = \langle \mathrm{J}_\beta[A], F\rangle$ and some $\eta \in \mathrm{Card}^N \cup \{\beta\}$, set $M := N\!\downarrow\!\eta$. Assume that $\pi : M \rightarrow_{\Sigma_0} Q$ is a cofinal map, Q some topless premouse. Then one can construct an ultrapower of N using π, yielding some map $\tilde{\pi} : N \rightarrow R$, such that if R is transitive, then R is a premouse, $\tilde{\pi} \supseteq \pi$, and $Q = R\!\downarrow\!\tilde{\eta}$, where $\tilde{\eta} = \mathrm{On} \cap Q = \sup \pi''\eta$. $\tilde{\pi}$ is then called the *fine upward extension* of π with respect to N, N is called *M-based*, and R is denoted by $\mathrm{Ult}^*(N, \pi)$.

5.15 Definition (4.6) Let $\vartheta > \omega$ be regular and let $\pi : \bar{H} \rightarrow_{\Sigma_\omega} \mathrm{H}_\vartheta$, where $\bar{H}$ is transitive. Let $Q \in \mathrm{rge}(\pi)$ be a topless mouse and set $M := \pi^{-1}(Q)$. Then $\pi \restriction M : M \rightarrow_{\Sigma_\omega} Q$ and for any $\alpha \leqslant \gamma := \mathrm{On} \cap M$, $\pi \restriction (M \downarrow \alpha) : M \downarrow \alpha \rightarrow_{\Sigma_0} Q \restriction (\sup \pi''\alpha)$ cofinally.

Let $\alpha \leqslant \gamma$, $M' := M \downarrow \alpha$. π is called *1-lousy at α with respect to M* if $\kappa := \kappa^{M'}$ is the largest cardinal in M' and there is a M'-based mouse N above κ such that $\mathrm{Ult}^*(N, \pi \restriction M')$ is *not* a mouse above $\pi(\kappa) = \kappa^{Q\downarrow(sup\pi''\alpha)}$.

π is called *2-lousy at α with respect to M* if there is a M'-based coarse mouse (or mouse above α) N such that, if $\tilde{\pi} : N \rightarrow \mathrm{Ult}^*(N, \pi \restriction M') = R$ is the upward extension of $\pi \restriction M' : M' \rightarrow Q \restriction \pi''\alpha$, then R is *not* a coarse mouse (or mouse above $\tilde{\pi}(\alpha)$).

[1] In keeping with [Sch96], although it is a somewhat unfortunate piece of terminology, for any premouse M, we will call any ξ such that there exists $\lambda < \xi$ with $O^{\bar{M}}(\lambda) \geqslant \xi$ *overlapped*. This is *not* the same as saying that the extender-sequence E^M is overlapping: this would only be the case were ξ measurable itself.

π is called 3-*lousy at* α *with respect to* M if there exists a stretched M'-based Σ_0-mouse N such that $\mathrm{Ult}(N, \pi \restriction M')$ is a stretched premouse but not a stretched Σ_0-mouse.

Finally, π is called *perfect with respect to* M if it is neither 1-, 2-, nor 3-lousy at α with respect to M for any $\alpha \leqslant \gamma$.

5.16 Lemma (4.7) *Let ϑ be a regular uncountable cardinal, and let $\pi : \bar{H} \to \mathrm{H}_\vartheta$ be perfect with respect to M, where $M \in \bar{H}$, $\bar{H}$ is transitive, and $Q := \pi(M)$ is a mouse such that κ^M is the largest cardinal of M. Suppose that $\pi \restriction M : M \to Q$ cofinally. Then*

- *i) if Q is presolid, then so is M, and*
- *ii) if Q is solid, then so is M.*

5.17 Lemma (4.8) *Let ϑ be a regular uncountable cardinal, $\pi : \bar{H} \to_{\Sigma_\omega} \mathrm{H}_\vartheta$ where $\bar{H}$ is transitive and ${}^\omega\bar{H} \subseteq \bar{H}$. Let $Q \in \mathrm{rge}(\pi)$ be a topless mouse. Then π is perfect with respect to $M := \pi^{-1}(Q)$.*

Finally, we conclude our summary of strong core model basics by some remarks on a covering lemma.

5.18 Definition (2.8) A *weasel* is an extender structure $W = \mathrm{J}[E]$ such that for every $\alpha \in \mathrm{On}$, $W|\alpha$ is a mouse. W is called *universal* if the fine coiteration of W with any coiterable premouse terminates. W is called *weakly universal* if the fine coiteration of W with any mouse terminates.

5.19 Definition (5.1) Let M be a mouse or a weasel. Let $\alpha < \mathrm{On} \cap M$. M is called *weakly full above* α if for all $\beta \in [\alpha, \mathrm{On} \cap M)$, if $N \supseteq M \downarrow \beta$ is a collapsing mouse for $M \downarrow \beta$ then $N \subseteq M$. M is *weakly full* if it is weakly full above 0.

Weakly universal weasels are weakly full, as are solid mice [Sch96, Lemma 5.8]. Also, strong mice are weakly full [Sch96, Lemma 5.9].

5.20 Definition (5.2) Let $M = \langle \mathrm{J}_\xi[\bar{E}], \emptyset \rangle$ be a strong topless mouse, and let Γ be a non-empty set or class of uncountable regular cardinals. A weasel $W = \mathrm{L}[E]$ is called Γ-*full over* M if

i) $\mathrm{L}[E]$ end-extends M, and W is weakly full above ξ.

ii) For any $\alpha \geqslant \xi$, if $E_\alpha \neq \emptyset$ is an extender at κ, α on W, then $\kappa > \xi$, E_α is countably complete, and $\mathrm{cf}(\kappa) \in \Gamma$.

iii) For any $\alpha > \xi$, if $\xi < \kappa := \kappa^{W\downarrow\alpha} < \alpha$, $W\downarrow(\kappa^{+W\downarrow\alpha})$ is presolid, $\mathrm{cf}(\kappa) \in \Gamma$, and F is a countably complete extender at κ, α on $W\downarrow\alpha$ such that $\mathrm{Ult}(W\downarrow\alpha, F)|\alpha = W\downarrow\alpha$ and $\langle W\downarrow\alpha, F\rangle$ is weakly amenable, then $F = E_\alpha$.

W is called *full over M* if it is Γ-full over M when Γ is the class of all uncountable regular cardinals. W is called Γ-*full* (or full) if it is Γ-full (or full) over $\emptyset$.

5.21 Lemma (5.3) *For any strong topless mouse $M = \langle \mathrm{J}_\xi[\bar{E}], \emptyset\rangle$ and any non-empty set or class Γ of uncountable regular cardinals there is a unique Γ-full weasel over M, denoted by $W^\Gamma(M)$.*

$\mathrm{K^c}$, the *countably complete core model*, is the unique Γ-full weasel over $\emptyset$, where Γ is the class of all uncountable regular cardinals.

5.22 Lemma (5.10) *Assume $\neg\,\mathrm{L}^{\mathrm{strong}}$. Let $\kappa \in \mathrm{Card}^{\mathrm{K^c}}$ be such that κ is not overlapped in $\mathrm{K^c}$. Let $\nu := O^{\mathrm{K^c}}(\kappa)$. Let $x \subseteq \nu$ be a set such that $\mathrm{card}(x)^{\aleph_0} < \mathrm{card}(\kappa)$. Then there is some $y \in \mathrm{K^c}$ such that $x \subseteq y$ and $\mathrm{card}^{\mathrm{K^c}}(y) \leqslant \kappa$.*

Chapter 6

Short Iterations

"Mine is a long and a sad tale!" said the Mouse, turning to Alice, and sighing.

"It is *a long tail, certainly," said Alice, looking down with wonder at the Mouse's tail; "but why do you call it sad?"*

(Lewis Carroll, Alice's Adventures in Wonderland, Chapter III)

In this chapter, we show how under considerably more restrictive assumptions than just the absence of an inner model for a strong cardinal one can give an upper bound for the length of certain short iterations. This will be done by considering an in a sense complete iteration into which any short iteration may be embedded.

Let us introduce the following definition:

6.1 DEFINITION Let $\bar{M}$ be a mouse, and let $\mathcal{I}$ be an iteration of $\bar{M}$, of length ϑ, with indices $\langle \nu_i \mid i < \vartheta \rangle$ and iteration maps $\langle \pi_{ij} \mid i \leqslant j \leqslant \vartheta \rangle$. $\mathcal{I}$ is *short* if no extender has been used in $\mathcal{I}$ for $\omega_1 + 1$ many times in a row, i. e., if for no $j < \vartheta$ with $\mathrm{cf}(j) = \omega_1$ the set

$$P(j, \nu_j) := \{i < j \mid \pi_{ij}(\nu_i) = \nu_j\}$$

is unbounded in j.

Given a mouse $\bar{M}$, we want to find an upper bound for the ordinal height of short iterates of $\bar{M}$.

6.2 Lemma *Let $\bar{M}$ be a sound, topless mouse,* $\mathrm{card}(\bar{M}) < \kappa$, *$\kappa$ some regular cardinal greater than ω_1. Assume that there is no inner model for $o^M(\mu) \geqslant \omega$. Then there is an ordinal $s < \kappa$, denoted by $s^{(\omega_1)}(\bar{M})$, which is an upper bound for the ordinal height of short iterates of $\bar{M}$:*

$$s^{(\omega_1)}(\bar{M}) \geqslant \sup\{(\mathrm{On} \cap M) + 1 \mid M \textit{ is a short, simple, fine iterate of } \bar{M}\}.$$

Proof We take $s^{(\omega_1)}$ to be the ordinal height of one specially defined iterate of $\bar{M}$. This iterate, $M^{\vartheta(\omega_1)}$, will be defined recursively. The lemma will be proved if we can show that any arbitrary short, simple, fine iterate M^* of $\bar{M}$ can be embedded into $M^{\vartheta(\omega_1)}$. Let us note that the iteration yielding $M^{\vartheta(\omega_1)}$ will, in fact, not be short itself.

For a mouse M, let

$$\begin{aligned} c(M) &:= \{\mu \in \mathrm{On} \cap M \mid \mu \text{ is the critical point of some extender } E^M_\nu\}, \\ d(M,\mu) &:= \{\nu \in \mathrm{On} \cap M \mid E^M_\nu \text{ is at } \mu,\, \nu\}. \end{aligned}$$

Note that we assumed $\bar{M}$ to be topless, so that these definitions really capture all possible extenders of $\bar{M}$.

Set $M^0 := \bar{M}$. Let $\Lambda = \omega_1^\omega$ (using ordinal exponentiation). Define bookkeeping functions m^0 and $\widetilde{m}^0 : c(M^0) \to \Lambda + 1$, setting $m^0(\xi) = \widetilde{m}^0(\xi) = 0$ for all ξ.

Let j be a limit ordinal and assume $\langle M^i \mid i < j\rangle$ had been defined, with iteration maps $\langle \pi_{ii'} \mid i \leqslant i' < j\rangle$. Then let M^j be the direct limit of this system, with corresponding maps $\langle \pi_{ij} \mid i < j\rangle$. Define the auxiliary bookkeeping function $\widetilde{m}^j$ as follows

$$\widetilde{m}^j(\xi) := \sup\{m^i(\bar{\xi}_i) \mid i < j \ \wedge\ \pi_{ij}(\bar{\xi}_i) = \xi\}$$

By induction, if all m^i take values less than $\Lambda + 1$, then so does $\widetilde{m}^j$. Note that if $\xi \in c(M^j)$ and $\xi = \pi_{ij}(\bar{\xi}_i)$, then $\bar{\xi}_i \in c(M^i)$ and hence $m^i(\bar{\xi}_i)$ is defined.

If j is a successor ordinal $i+1$, then set

$$\widetilde{m}^j(\xi) := \begin{cases} m^i(\bar{\xi}_i) & \text{if } \xi = \pi_{ij}(\bar{\xi}_i) \\ 0 & \text{if } \xi \notin \mathrm{rge}(\pi_{ij}). \end{cases}$$

Again, note that if $\xi \in c(M^j) \cap \mathrm{rge}(\pi_{ij})$, then $m^i(\bar{\xi}_i)$ is defined.

To construct M^{j+1}, for arbitrary j, let first be

$$\mu_j := \text{the least } \mu \in c(M^j) \text{ such that } \widetilde{m}^j(\mu) < \Lambda \text{ and } \forall i < j \ (\nu_i < \mu).$$

We are thus looking for the first critical point which has not been used up Λ-many times yet, and which would allow us to continue the iteration in a normal way.

Case 0 If no such μ exists, then set $\vartheta^{(\omega_1)} := j$ and terminate the construction.

Case 1 $\mathrm{cf}(j) \neq \omega_1$ (including the case that j is a successor ordinal).

Let $\nu_j := \min d(M^j, \mu_j)$, and set $M^{j+1} := \mathrm{Ult}(M^j, E^{M^j}_{\nu_j})$, $\pi_{j,j+1} :=$ the canonical embedding, and for $i < j$ let $\pi_{i,j+1} := \pi_{j,j+1} \circ \pi_{ij}$. Set $m^j := \widetilde{m}^j$. I. e., we take the least extender associated with this critical point and take the ultrapower with it. The bookkeeping function remains unchanged.

Case 2 $\mathrm{cf}(j) = \omega_1$.

Let ν_j be the least $\nu \in d(M^j, \mu_j)$ such that the set

$$P(j, \nu) := \{ i < j \mid \pi_{ij}(\nu_i) = \nu \}$$

is bounded in j.

Case 2.1 If such ν_j exists, again set $M^{j+1} := \mathrm{Ult}(M^j, E^{M^j}_{\nu_j})$, $\pi_{j,j+1} :=$ the canonical embedding, $\pi_{i,j+1}$ accordingly, and $m^j := \widetilde{m}^j$. Thus instead of just taking the least extender with the chosen critical point, we insist that it had also only been used boundedly often before j.

Case 2.2 If no such ν exists, i. e., if for all $\nu \in d(M^j, \mu_j)$, $P(j, \nu)$ is *unbounded* in j, then set $\nu_j := \min d(M^j, \mu_j)$, and let $M^{j+1} := \mathrm{Ult}(M^j, E^{M^j}_{\nu_j})$. In this case, we have exhausted all extenders with critical point μ_j, i. e., used them all cofinally up to j, and now start all over again. Informally, we will call this a *block* of the iteration. To keep track of how often we do this, we now increase the bookkeeping function at μ_j, setting

$$m^j(\xi) := \begin{cases} \widetilde{m}^j(\xi) + 1 & \text{if } \xi = \mu_j \\ \widetilde{m}^j(\xi) & \text{else.} \end{cases}$$

This completes the definition of the iteration.

Claim 1 Let $j < \vartheta^{(\omega_1)}$, and assume $\widetilde{m}^j(\mu) < \Lambda$. Then

$$\forall \mu' \geqslant \mu \; (\widetilde{m}^j(\mu') < \Lambda),$$

i. e., not only is μ_j the least μ such that $\widetilde{m}^j(\mu) < \Lambda$, it is also the least upper bound (in $c(M^j)$) of those μ that have $\widetilde{m}^j(\mu) = \Lambda$ (as well as satisfying the normality requirement $\forall i < j \; (\nu_i < \mu)$). Since $m^j(\xi) \leqslant \widetilde{m}^j(\xi) + 1$ for any ξ, this claim is also true if one replaces $\widetilde{m}^j$ by m^j.

Proof Assume the claim were false. Let j be the least index such that this happens. Recall that μ_j was defined to be the least μ such that $\widetilde{m}^j(\mu) < \Lambda$. Thus there must be some $\mu > \mu_j$, $\mu \in c(M^j)$, such that $\widetilde{m}^j(\mu) = \Lambda$. As j was chosen minimally,

$$\forall i < j \; \Big(\mu_i = \min c(M^i) \setminus \{\mu' \in c(M^i) \mid \widetilde{m}^i(\mu') = \Lambda \;\wedge\; \forall i' < i \; (\nu_{i'} < \mu')\}\Big).$$

First note that j must be a limit ordinal. Assume to the contrary that $j = i + 1$. Then we must have $\mu \in \mathrm{rge}(\pi_{ij})$, as otherwise $\widetilde{m}^j(\mu) = 0$. So let $\mu = \pi_{ij}(\bar{\mu})$. Obviously, $\bar{\mu} \geqslant \mu_i$, as otherwise $\mu = \pi_{ij}(\bar{\mu}) = \bar{\mu} < \mu_i < \mu_j < \mu$, contradiction. But then, by the minimal choice of j, $\widetilde{m}^i(\bar{\mu}) < \Lambda$, whence $m^i(\bar{\mu}) < \Lambda$, as well, so that $\widetilde{m}^j(\mu) = m^i(\bar{\mu}) < \Lambda$. Contradiction.

So j is a limit ordinal. Find some $i_0 < j$ such that $\mu, \mu_j \in \mathrm{rge}(\pi_{i_0 j})$, and set $\bar{\mu}_i := \pi_{ij}^{-1}(\mu)$ and $\tilde{\mu}_i := \pi_{ij}^{-1}(\mu_j)$, for $i \in [i_0, j)$. As $\Lambda > \xi_0 := \widetilde{m}^j(\mu_j) = \sup\{m^i(\tilde{\mu}_i) \mid i < j\}$, we know that

$$\forall i \in [i_0, j) \; (m^i(\tilde{\mu}_i) \leqslant \xi_0 < \Lambda).$$

Note that again we must have that for all $i \in [i_0, j)$ $(\bar{\mu}_i > \tilde{\mu}_i \geqslant \mu_i)$: the first inequality is immediate from $\mu > \mu_j$, whereas the failure of the second would imply that $\mu_j = \pi_{ij}(\tilde{\mu}_i) = \tilde{\mu}_i < \mu_i < \mu_j$, a contradiction. But then we have that

$$\forall i \in [i_0, j)\ (m^i(\bar{\mu}_i) = \widetilde{m}^i(\bar{\mu}_i)),$$

as this holds for all μ', unless we are in Case 2.2 of the definition of the iteration and $\mu' = \mu_i$ (whereas we just showed that $\bar{\mu}_i > \mu_i$). Thus inductively ones sees that

$$\forall i \in [i_0, j)\ (m^i(\bar{\mu}_i) = \widetilde{m}^{i_0}(\bar{\mu}_{i_0})).$$

As the claim is true at i_0, we conclude that

$$m^{i_0}(\bar{\mu}_{i_0}) < \Lambda$$

since $\bar{\mu}_{i_0} > \tilde{\mu}_{i_0}$ and $m^{i_0}(\tilde{\mu}_{i_0}) \leqslant \xi_0 < \Lambda$. Thus

$$\widetilde{m}^j(\mu) = \sup\{m^i(\bar{\mu}_i) \mid i \in [i_0, j)\} = m^{i_0}(\bar{\mu}_{i_0}) < \Lambda,$$

contradicting the assumption. □(Claim 1)

Claim 2 The construction terminates at some $\vartheta^{(\omega_1)} < \kappa$.

Proof Assume to the contrary that $\vartheta^{(\omega_1)} \geqslant \kappa$. We will show first that the set

$$C := \{j \in \kappa \mid \mathrm{cf}(j) = \omega_1 \text{ and } P(j) := P(j, \nu_j) \text{ is unbounded in } j\}$$

is stationary in κ. Assume to the contrary that there is some club subset $D \subseteq \kappa$ such that $C \cap D = \emptyset$. Let $C_0 := \kappa \setminus C$ and $C_1 := C_0 \cap \{j \in \kappa \mid \mathrm{cf}(j) = \omega_1\}$. Then C_1 is stationary in κ: let D' be a club subset of κ. We have to show $C_1 \cap D' \neq \emptyset$. Since both D and D' are club, so is $D \cap D'$. Let i be a limit point of $D \cap D'$ of cofinality ω_1. As $i \in D$ and $D \subseteq C_0$, we must have $i \in C_1$. Thus $i \in C_1 \cap D'$, and so C_1 must, indeed, be stationary.

Let

$$\begin{aligned} E := \{j \in \kappa \mid\ & j \text{ is a limit ordinal and} \\ & j \text{ is closed under } \Gamma \text{ and } \Gamma_3 \text{ and } (i < j \ \rightarrow\ \mathrm{On} \cap M^i < j)\}, \end{aligned}$$

where Γ denotes the Gödel Pairing Function and Γ_3 the derived function for triples. Then E is club in κ. Hence $C_2 := C_1 \cap E$ is stationary in κ. Define a function $f : C_2 \to \kappa$ by letting

$$f(j) := \Gamma_3(i, \nu, b),$$

where i is the least i such that $\nu_j \in \mathrm{rge}(\pi_{ij})$, $\nu = \pi_{ij}^{-1}(\nu_j)$, and b is the least upper bound of $P(j)$. Note that $i < j$ as j is a limit ordinal, $\nu < j$ as $\mathrm{On} \cap M^i < j$ ($j \in E$!), and $b < j$ as $j \in C_1$. Thus, as j is closed under Gödel Pairing, $f(j) < j$, i. e., f is regressive. By Fodor's Theorem, there is a stationary set $C_3 \subseteq C_2$, and $\bar{\imath}$, $\bar{\nu}$, and $\bar{b}$ auch that

$$\forall j \in C_3 \ (f(j) = \Gamma_3(\bar{\imath}, \bar{\nu}, \bar{b})).$$

Now choose $i < j \in C_3$. Then $\pi_{\bar{\imath}i}(\bar{\nu}) = \nu_i$ and $\pi_{ij}(\nu_i) = \pi_{ij}(\pi_{\bar{\imath}i}(\bar{\nu})) = \pi_{\bar{\imath}j}(\bar{\nu}) = \nu_j$. Thus $i \in P(j)$, so that $i < \bar{b} =$ the least upper bound of $P(j)$. On the other hand, $i > \bar{b}$, as $\bar{b}$ is also the least upper bound of $P(i)$, which must be less than i as $i \in C_0$. This is a contradiction. Hence C really is stationary in κ.

Assume w.l.o.g. that $C \subseteq E$. Now again define a function $f : C \to \kappa$ by letting

$$f(j) := \Gamma(i, \nu)$$

where i and ν are chosen as before. Again, there will be a stationary set $C_4 \subseteq C$ and $\bar{\imath}$, $\bar{\nu}$ such that

$$\forall j \in C_4 \ (f(j) = \Gamma(\bar{\imath}, \bar{\nu})).$$

Pick a sequence $\langle j_\xi \mid \xi \leqslant \Lambda \rangle$ of elements of C_4. Thus for all $\xi \leqslant \Lambda$, $P(j_\xi)$ is unbounded in j_ξ. Note that this can only happen if we are in Case 2.2 of the definition of the iteration. (Not only is $P(j_\xi) = P(j_\xi, \nu_{j_\xi})$ unbounded in j_ξ but one also uses the extender $E_{\nu_{j_\xi}}^{M^{j_\xi}}$ to construct the next ultrapower.) In this case, the bookkeeping function m^{j_ξ} is incremented by one at μ_{j_ξ}. Thus inductively one sees that

$$\forall \xi < \Lambda (m^{j_\xi}(\mu_{j_\xi}) \geqslant \xi).$$

and so

$$m^{j_\Lambda}(\mu_{j_\Lambda}) \geqslant \sup\{m^{j_\xi}(\mu_{j_\xi}) \mid \xi < \Lambda\} + 1 \geqslant \Lambda + 1$$

which is absurd, as no m^j ever takes values greater than Λ. (Note that, of course, if $\pi_{ij}(\nu_i) = \nu_j$, then $\pi_{ij}(\mu_i) = \mu_j$, too.) Thus the iteration cannot have had length $\vartheta^{(\omega_1)} \geqslant \kappa$. $\square$(Claim 2)

Claim 3 Let $N^* = N^\vartheta$ be a short, simple, fine, normal iterate of $N^0 := \bar{M}$, with indices $\langle \eta_j \mid j < \vartheta\rangle$, critical points $\langle \lambda_j \mid j < \vartheta\rangle$, and maps $\langle \pi^N_{ij} \mid i \leqslant j \leqslant \vartheta\rangle$. Then there exists an embedding $h : N^* \to M^{\vartheta^{(\omega_1)}}$, and hence $\mathrm{On} \cap N^* \leqslant \mathrm{On} \cap M^{\vartheta^{(\omega_1)}}$.

Proof We will inductively construct a sequence of maps h^j, $j < \vartheta$, such that the limit of this sequence will be the desired map h. Each h^j will map N^j into some M^{α_j}. Set $\alpha_0 := 0$, $h^0 := \mathrm{id}\big|_{\bar{M}}$.

Basically, if we have embedded N^j into M^{α_j} and are given the task of embedding $N^{j+1} = \mathrm{Ult}(N^j, E^{N^j}_{\eta_j})$ into some $M^{\alpha_{j+1}}$, first consider the image of $E^{N^j}_{\eta_j}$ under h^j. Set $\widehat{\eta}_j := h^j(\eta_j)$ and $\widehat{\lambda}_j := h^j(\lambda_j)$, where $\lambda_j = \mathrm{crit}(E^{N^j}_{\eta_j})$. By elementarity of h^j, $E^{M^{\alpha_j}}_{\widehat{\eta}_j}$ will be an extender at $\widehat{\lambda}_j$, $\widehat{\eta}_j$. Now ν_{α_j} will not necessarily be equal to $\widehat{\eta}_j$. But possibly there will be a (least) $\beta_j \geqslant \alpha_j$ such that $\pi^M_{\alpha_j\beta_j}(\widehat{\eta}_j) = \nu_{\beta_j}$. In this case, let $g^j := \pi^M_{\alpha_j\beta_j} \circ h^j$. We can then use [Koe89, Theorem 14.2] to get the desired map $h^{j+1} : N^{j+1} \to M^{\alpha_{j+1}}$, where $\alpha_{j+1} := \beta_j + 1$, completing the diagramme in Figure 6.1.

$$\begin{tikzcd}
M^{\alpha_j} \arrow[r, "\pi^M_{\alpha_j\beta_j}"'] & M^{\beta_j} \arrow[r, "E^{M^{\beta_j}}_{\nu_{\beta_j}}"'] & M^{\beta_j+1} \\
N^j \arrow[u, "h^j"] \arrow[ur, "g^j"'] \arrow[rr, "\pi^N_{j,j+1}"', "E^{N^j}_{\eta_j}"'] & & N^{j+1} \arrow[u, dashed, "h^{j+1}"']
\end{tikzcd}$$

Figure 6.1: The map h^{j+1} is the canonical completion of the diagramme.

At limits, the obvious completion is used. The problem thus reduces to ensuring that at every stage $j < \vartheta$, there exists some $\beta_j \geqslant \alpha_j$ such that

$\pi^M_{\alpha_j\beta_j}(\widehat{\eta}_j) = \nu_{\beta_j}$, so that we can construct h^{j+1}. Furthermore, we have to do some bookkeeping. Inductively, we prove the following statement

$$(*)\qquad \begin{array}{ll} (1) & \forall\lambda \geqslant \lambda_j\ \left(\tilde{n}^j(\lambda) = \widetilde{m}^{\alpha_j}(h^j(\lambda))\right) \\ (2) & \forall\lambda \geqslant \lambda_j\ \left(n^j(\lambda) = m^{\alpha_j}(h^j(\lambda)) = m^{\beta_j}(g^j(\lambda))\right), \end{array}$$

where the functions $\tilde{n}^j$ and n^j will be defined similarly to $\widetilde{m}^j$ and m^j.

Case 1 j is a successor ordinal, $j = i + 1$.

Set

$$\tilde{n}^j(\xi) := \begin{cases} n^i(\bar{\xi}_i) & \text{if } \xi = \pi^N_{ij}(\bar{\xi}_i) \\ 0 & \text{if } \xi \notin \operatorname{rge}(\pi^N_{ij}). \end{cases}$$

We claim that $\mu_{\alpha_j} \leqslant \widehat{\lambda}_j$. We first show that the normality requirement is fulfilled, i. e., that

$$\forall\alpha < \alpha_j\ (\nu_\alpha < \widehat{\lambda}_j).$$

It is of course sufficient to show $\nu_{\beta_i} < \widehat{\lambda}_j$, as $\alpha_j = \beta_i + 1$ and $\mathcal{I}^M$ is normal. Assume to the contrary that $\widehat{\lambda}_j \leqslant \nu_{\beta_i}$. By construction of M^{β_i+1} we have that $o^{M^{\beta_i+1}}(\mu_{\beta_i}) = \nu_{\beta_i}$. Since we are in a situation where extender sequences do not overlap, we conclude that then $\widehat{\lambda}_j \leqslant \mu_{\beta_i}$, as by elementarity of h^j, $\widehat{\lambda}_j$ is measurable in M^{β_i+1}. Using the fact that g^i and h^j agree up to η_i we now reach the following contradiction:

$$\widehat{\lambda}_j \leqslant \mu_{\beta_i} = g^i(\lambda_i) = h^j(\lambda_i) < h^j(\lambda_j) = \widehat{\lambda}_j.$$

By Claim 1, the bookkeeping function m^{β_i} takes values less than Λ for arguments greater than or equal to $\mu_{\beta_i} = \pi^M_{\alpha_i\beta_i}(\widehat{\lambda}_i)$. But at successor steps (in the iteration of M) the bookkeeping function does not increase, so that

$$\forall\xi \geqslant \mu_{\beta_i}\ (m^{\beta_i+1}(\xi) < \Lambda)$$

This is true for those $\xi \in \operatorname{rge}(\pi^M_{\beta_i,\beta_{i+1}})$, $\xi \geqslant \pi^M_{\beta_i,\beta_{i+1}}(\mu_{\beta_i})$, as they keep the m-values of their pre-images, as well as for those $\xi \notin \operatorname{rge}(\pi^M_{\beta_i,\beta_{i+1}})$, as they take m-value 0 by definition. (We are not interested in ξ less than μ_{β_i}. In fact, we are not even interested in ξ less than ν_{β_i}, as we have $\widehat{\lambda}_j > \nu_{\beta_i}$. The m-values for these ξ might indeed be Λ. In fact, they are bound to be Λ, as otherwise μ_{β_i} would not have been chosen correctly!)

Now, since $\widehat{\lambda}_j > \mu_{\beta_i}$, we conclude that

$$m^{\alpha_j}(\widehat{\lambda}_j) < \Lambda.$$

We know inductively that $\alpha_j = \alpha_{i+1} = \beta_i + 1$, i. e., α_j is itself a successor ordinal. Hence $P^M(\alpha_j, \widehat{\eta}_j)$ is bounded in j (by i, trivially). Thus $\nu_{\alpha_j} \leqslant \widehat{\eta}_j$. This implies that the extender $E^{M^{\alpha_j}}_{\widehat{\eta}_j}$ *will* be used later on in the iteration, i. e., there does exist some least $\beta_j \geqslant \alpha_j$ such that $\pi^M_{\alpha_j\beta_j}(\widehat{\eta}_j) = \nu_{\beta_j}$.

Thus we can now set $\alpha_{j+1} := \beta_j + 1$ and define $h^{j+1} : N^{j+1} \to M^{\alpha_{j+1}}$ as described above. Set $n^j := \widetilde{n}^j$

Case 2 j is a limit ordinal.

Note that $\mathrm{cf}(\alpha_j) = \mathrm{cf}(j)$. Set

$$\widetilde{n}^j(\xi) := \sup\{n^i(\bar{\xi}_i) \mid i < j \ \wedge \ \pi^N_{ij}(\bar{\xi}_i) = \xi\}$$

Case 2.1 $\mu_{\alpha_j} \leqslant \widehat{\lambda}_j$.

We must have $m^{\alpha_j}(\widehat{\lambda}_j) < \Lambda$, by Claim 1.

Case 2.1.1 $\mathrm{cf}(j) \neq \omega_1$.

As $\mathrm{cf}(\alpha_j) \neq \omega_1$, there will be a least $\beta_j \geqslant \alpha_j$ such that $\pi^M_{\alpha_j\beta_j}(\widehat{\eta}_j) = \nu_{\beta_j}$.

As in Case 1, set $\alpha_{j+1} := \beta_j + 1$ and define $h^{j+1} : N^{j+1} \to M^{\alpha_{j+1}}$ as described above. Note that we must have $m^{\alpha_j} = \widetilde{m}^{\alpha_j}$, as this is true for all α except those of cofinality ω_1, in the special Case 2.2 of the definition of the iteration of M. Set $n^j := \widetilde{n}^j$.

Case 2.1.2 $\mathrm{cf}(j) = \omega_1$.

Case 2.1.2.1 $\nu_{\alpha_j} > \widehat{\eta}_j$.

In this case, we first go to M^{α_j+1}. Note that there, $P^M(\alpha_j+1, \pi^M_{\alpha_j,\alpha_j+1}(\widehat{\eta}_j))$ will be bounded in α_j+1 (by α_j, trivially). Also, $\mu_{\alpha_j+1} \leqslant \pi^M_{\alpha_j,\alpha_j+1}(\mu_{\alpha_j})$, as this last term satisfies both conditions for the definition of μ_{α_j+1}: $\widetilde{m}^{\alpha_j+1}(\pi^M_{\alpha_j,\alpha_j+1}(\mu_{\alpha_j}))$ $= m^{\alpha_j}(\mu_{\alpha_j})$ by definition, and this last term must be less than Λ, by the choice of μ_{α_j}. But then there will be some least $\beta_j \geqslant \alpha_j+1$ such that $\nu_{\beta_j} = \pi^M_{\alpha_j,\beta_j}(\widehat{\eta}_j)$ and we are done. Set $n^j := \tilde{n}^j$.

Case 2.1.2.2 $\nu_{\alpha_j} \leqslant \widehat{\eta}_j$ and $P^M(\alpha_j, \nu_{\alpha_j})$ is bounded in α_j.

In this case the existence of a least suitable β_j is immediate. Set $n^j := \tilde{n}^j$.

Case 2.1.2.3 $\nu_{\alpha_j} \leqslant \widehat{\eta}_j$ and $P^M(\alpha_j, \nu_{\alpha_j})$ is unbounded in α_j.

In this case we have reached, on the M-side of the iteration, a point where the critical point μ_{α_j} has been exhausted, i. e., we are in the situation of Case 2.2 of the definition of the iteration of M. For in no other case will an extender $E^{M^\alpha}_\nu$ be used although $P^M(\alpha, \nu)$ is unbounded in α and $\mathrm{cf}(\alpha) = \omega_1$. Note that this implies that *for all* $\nu \in d(M^{\alpha_j}, \mu_{\alpha_j})$ we must have that $P^M(\alpha_j, \nu)$ is unbounded in α_j, and that $\nu_{\alpha_j} = \min d(M^{\alpha_j}, \mu_{\alpha_j})$.

However, there will none the less exist some least $\beta_j \geqslant \alpha_j$ such that $\nu_{\beta_j} = \pi^M_{\alpha_j,\beta_j}(\widehat{\eta}_j)$, so that the construction of h^{j+1} can proceed as before. The only difference is that we have to adjust the bookkeeping function, provided that $h^j(\lambda_j) = \mu_{\alpha_j}$. In this case, set

$$n^j(\xi) := \begin{cases} \tilde{n}^j(\xi) + 1 & \text{if } \xi = \lambda_j \\ \tilde{n}^j(\xi) & \text{else,} \end{cases}$$

Note that $m^{\alpha_j}(\mu_{\alpha_j})$ increases by one at this point, too. Otherwise (i. e., if $h^j(\lambda_j) > \mu_{\alpha_j}$) let $n^j = \tilde{n}^j$. The following subclaim will be used later on.

Subclaim If $h^j(\lambda_j) = \mu_{\alpha_j}$, then the set $Q^N(j) := \{i < j \mid \pi^N_{ij}(\lambda_i) = \lambda_j\}$ is unbounded in j.

This says that we can conclude that one and the same critical point must have been used cofinally up to j on the N-side. Unfortunately, we cannot tell which of the extenders (with this critical point) was responsible, i. e., we cannot show that $P^N(j, \eta_j)$ is unbounded in j, as well. However, by later on taking sufficiently closed limits, we will be able to remedy the situation.

Proof j is a limit ordinal, so there exists some $i_0 < j$ such that $\lambda_j \in \mathrm{rge}(\pi^N_{i_0 j})$. For $i \in [i_0, j)$, let $\bar{\lambda}_i := \left(\pi^N_{ij}\right)^{-1}(\lambda_j)$. Note that then $\bar{\lambda}_i \geqslant \lambda_i$, as otherwise $\lambda_j = \pi^N_{ij}(\bar{\lambda}_i) = \bar{\lambda}_i < \lambda_i$, contradicting the normality of $\mathcal{I}^N$. As a consequence, we can conclude that $\pi^N_{ij}(\lambda_i) \leqslant \lambda_j$. Also, set $\bar{\mu}_\alpha := \left(\pi^M_{\alpha\alpha_j}\right)^{-1}(h^j(\lambda_j))$, for $\alpha \geqslant \alpha_{i_0}$, so that $\bar{\mu}_{\alpha_i} = h^i(\bar{\lambda}_i)$.

Now assume for a contradiction that $Q^N(j)$ were bounded in j. Then there exists some i_1 (w.l.o.g. $i_1 \geqslant i_0$) such that for all $i \in [i_1, j)$ $(\pi^N_{ij}(\lambda_i) < \lambda_j)$, i. e., $\lambda_i < \bar{\lambda}_i$. Subsequently,

$$\forall i \in [i_1, j)\ (h^i(\lambda_i) < \bar{\mu}_{\alpha_i}).$$

Since $P^M(\alpha_j, h^j(\eta_j))$ is unbounded in α_j, it follows that also $Q^M(\alpha_j, h^j(\lambda_j)) := \{\alpha < \alpha_j \mid \pi^M_{\alpha\alpha_j}(\mu_\alpha) = h^j(\lambda_j)\}$ is unbounded in α_j. (If some extender is used cofinally often, then, a fortiori, its critical point is used cofinally often.) Recall that $\alpha_i \leqslant \beta_i$ and $\mu_{\beta_i} = \pi^M_{\alpha_i\beta_i}(h^i(\lambda_i))$. Thus

$$\mu_{\beta_i} < \pi^M_{\alpha_i\beta_i}(h^i(\bar{\lambda}_i)) = \bar{\mu}_{\beta_i}.$$

Now for $\alpha \in Q^M(\alpha_j, h^j(\lambda_j))$, $\mu_\alpha = \bar{\mu}_\alpha$. Thus it cannot hold that for all sufficiently large i, $\alpha_i = \beta_i$. For then we would have that for all sufficiently large α, there exists some i such that $\alpha = \alpha_i = \beta_i$, and thus $\mu_\alpha = \mu_{\beta_i} < \bar{\mu}_{\beta_i}$. This would imply $\alpha \notin Q^M(\alpha_j, h^j(\eta_j))$, so that this set would be bounded in α_j, contradiction. This is just a complicated way of saying that if on the M-side the critical point corresponding to λ_j is used cofinally often below α_j, and if the upwards maps h^i from N^i always require the use of critical points corresponding to $\lambda_i < \bar{\lambda}_i = \left(\pi^N_{ij}\right)^{-1}(\lambda_j)$, then sometimes we have to skip from α_i to β_i to make room for possible uses of the higher critical point (cf. Figure 6.2).

$$\begin{tikzcd}
M^{\alpha_i} \arrow[r] & \cdots \arrow[r] & M^{\alpha} \arrow[r, "\mathrm{crit}=\bar{\mu}_\alpha"'] & M^{\alpha+1} \arrow[r] & \cdots \arrow[r] & M^{\beta_i} \arrow[r, "\mathrm{crit}=\mu_{\beta_i}"'] & M^{\beta_i+1} = M^{\alpha_{i+1}} \\
N^i \arrow[u, "h^i"] \arrow[urrrrr, "g^i"'] \arrow[rrrrrr, "\mathrm{crit}=\lambda_i<\bar{\lambda}_i"'] & & & & & & N^{i+1} \arrow[u, "h^{i+1}"']
\end{tikzcd}$$

Figure 6.2: α_i cannot always be equal to β_i if the pre-image of λ_j is not used cofinally below j.

But this will yield a contradiction. Pick some $i \in [i_1, j)$ such that $\alpha_i < \beta_i$, and there is $\alpha \in [\alpha_i, \beta_i) \cap Q^M(\alpha_j, h^j(\lambda_j))$, i. e., $\mu_\alpha = \bar{\mu}_\alpha$. Then

$$\begin{aligned}\pi^M_{\alpha_i\alpha}(h^i(\lambda_i)) &< \pi^M_{\alpha_i\alpha}(h^i(\bar{\lambda}_i))\\ &= \pi^M_{\alpha_i\alpha}(\bar{\mu}_{\alpha_i})\\ &= \bar{\mu}_\alpha = \mu_\alpha = \mathrm{crit}(\pi^M_{\alpha,\alpha+1}) < \mathrm{crit}(\pi^M_{\alpha+1,\beta_i})\end{aligned}$$

so that

$$\begin{aligned}\pi^M_{\alpha_i\beta_i}(h^i(\lambda_i)) &= \pi^M_{\alpha+1,\beta_i} \circ \pi^M_{\alpha,\alpha+1} \circ \pi^M_{\alpha_i\alpha}(h^i(\lambda_i))\\ &= \pi^M_{\alpha+1,\beta_i}\Big(\pi^M_{\alpha_i\alpha}(h^i(\lambda_i))\Big)\\ &= \pi^M_{\alpha_i\alpha}(h^i(\lambda_i))\\ &< \mu_\alpha,\end{aligned}$$

while at the same time

$$\pi^M_{\alpha_i\beta_i}(h^i(\lambda_i)) = \mu_{\beta_i} > \mu_\alpha,$$

by the normality of $\mathcal{I}^M$. Contradiction. □(Subclaim)

Case 2.2 $\mu_{\alpha_j} > \widehat{\lambda}_j$ or $\alpha_j = \vartheta^{(\omega_1)}$.

In this case, no suitable β_j exists. Let the construction break down.

□(Construction of h^j)

We now turn to the proof of $(*)$. Properly, this proof should be part of the same induction, but to avoid confusion, we separate the two. Say $(*)$ had been shown to hold for all $i < j$.

Case 1 $j = i + 1$ a successor ordinal.

Equation (1) "at j" follows from (2) "at i": Let $\lambda \geqslant \lambda_j$. If $\lambda \in \mathrm{rge}(\pi^N_{ij})$, say $\lambda = \pi^N_{ij}(\bar{\lambda})$, then

$$h^j(\lambda) = h^j(\pi^N_{ij}(\bar{\lambda})) = \pi^M_{\beta_i,\beta_i+1}(g^i(\bar{\lambda})).$$

Thus $\widetilde{m}^{\alpha_j}(h^j(\lambda)) = m^{\beta_i}(g^i(\bar{\lambda})) = n^i(\bar{\lambda}) = \tilde{n}^j(\lambda)$, applying the definition of $\widetilde{m}^{\alpha_j}$, $(*)$ (2) at i, and finally the definition of $\tilde{n}^j$. If $\lambda \notin \mathrm{rge}(\pi^N_{ij})$, then $\tilde{n}^j(\lambda) = 0$, by definition. It suffices to show that $h^j(\lambda) \notin \mathrm{rge}(\pi^M_{\beta_i,\beta_i+1})$ as well, as then

$\widetilde{m}^{\alpha_j}(h^j(\lambda)) = 0$, again by definition. So assume to the contrary that $h^j(\lambda) = \pi^M_{\beta_i,\beta_i+1}(\bar{\xi})$. Now $\lambda = \pi^N_{ij}(f)(a)$ for some $a \in [\mu_i]^{<\omega}$, $f : [\mu_i]^{\bar{\bar{a}}} \to N^i$, and by the construction of h^j (cf. [Koe89, Theorem 14.2]), $h^j(\lambda) = \pi^M_{\beta_i,\beta_i+1}(g^i(f))(g^i(a))$. Now

$$N^i \vDash f \text{ is not constant modulo } E_a,$$

where $E := E^{N^i}_{\eta_i}$, whence

$$M^{\beta_i} \vDash g^i(f) \text{ is not constant modulo } \bar{E}_a,$$

where $\bar{E} := E^{M^{\beta_i}}_{\nu_{\beta_i}}$. But then $h^j(\lambda) = \pi^M_{\beta_i,\beta_i+1}(g^i(f))(g^i(a))$ is not an element of $\mathrm{rge}(\pi^M_{\beta_i,\beta_i+1})$ either. So $\widetilde{m}^{\alpha_j}(h^j(\lambda)) = 0 = \tilde{n}^j(\lambda)$.

This proves $(*)$ (1) "at j". As for (2), note that $\alpha_j = \beta_i + 1$ is a successor ordinal, whence $m^{\alpha_j} = \widetilde{m}^{\alpha_j}$, and that we set $n^j = \tilde{n}^j$. So it remains to show that $m^{\beta_j}(g^j(\lambda)) = m^{\alpha_j}(h^j(\lambda))$ for all $\lambda \geqslant \lambda_j$. But if this were not true, then we would have, somewhere between α_j and β_j, exhausted a critical point greater than or equal to $h^j(\lambda)$. But this is absurd, since then we would have used $E^{M^{\alpha_j}}_{h^j(\lambda)}$ not only once but in fact many times before β_j, contradicting the choice of β_j as the first index where this extender is used.

Case 2 j is a limit ordinal.

As in Case 1, $(*)$ (1) follows from (2) at earlier stages: Let $\lambda \geqslant \lambda_j$. Then $\lambda = \pi^N_{ij}(\bar{\lambda})$, for some $i < j$ and some $\bar{\lambda}$. Thus $h^j(\lambda) = \pi^M_{\alpha_i\alpha_j}(h^i(\bar{\lambda})) = \pi^M_{\beta_i\alpha_j}(g^i(\bar{\lambda}))$. Note that $\alpha_j = \sup\{\alpha_i \mid i < j\} = \sup\{\beta_i \mid i < j\}$, so that using (2) at i, one concludes that

$$\begin{aligned}
\widetilde{m}^{\alpha_j}(h^j(\lambda)) &= \sup\{m^{\alpha}(\bar{\xi}) \mid \alpha < \alpha_j \wedge \pi^M_{\alpha\alpha_j}(\bar{\xi}) = h^j(\lambda)\} \\
&= \sup\{m^{\beta_i}(\bar{\xi}) \mid i < j \wedge \pi^M_{\beta_i\alpha_j}(\bar{\xi}) = h^j(\lambda)\} \\
&= \sup\{m^{\beta_i}(\bar{\xi}) \mid i < j \wedge \bar{\xi} = g^i(\bar{\lambda}) \wedge \pi^N_{ij}(\bar{\lambda}) = \lambda\} \\
&= \sup\{n^i(\bar{\lambda}) \mid i < j \wedge \pi^N_{ij}(\bar{\lambda}) = \lambda\} \\
&= \tilde{n}^j(\lambda)
\end{aligned}$$

To see that (2) holds, note that $n^j(\lambda) = m^{\alpha_j}(h^j(\lambda))$ for all $\lambda \geqslant \lambda_j$. For $\lambda > \lambda_j$ this follows from $n^j(\lambda) = \tilde{n}^j(\lambda) = \widetilde{m}^{\alpha_j}(h^j(\lambda)) = m^{\alpha_j}(h^j(\lambda))$, noting that then $h^j(\lambda) > \mu_{\alpha_j}$, too. It is also true for λ_j itself, as $n^j(\lambda_j) = \tilde{n}^j(\lambda_j)$, unless we are in Case 2.1.2.3 and $h^j(\lambda_j) = \mu_{\alpha_j}$. But in this case $n^j(\lambda_j) = \tilde{n}^j(\lambda_j) + 1 = \widetilde{m}^{\alpha_j}(\mu_{\alpha_j}) + 1 = m^{\alpha_j}(\mu_{\alpha_j})$, so (2) holds anyway.

It remains to show that $m^{\beta_j}(g^j(\lambda)) = m^{\alpha_j}(h^j(\lambda))$. But β_j was chosen to be the least β such that $\nu_\beta = \pi^M_{\alpha_j,\beta}(h^j(\eta_j))$, i. e., the first place in the iteration of M where the extender $E^{M^{\alpha_j}}_{\widehat{\eta}_j}$ is used. This will happen long before the next point at which this critical point will be exhausted and hence m would be increased.

Note that (1) remains true when the construction of h^j breaks down in Case 2.2. □(Proof of $(*)$)

Subclaim The construction goes through for all $j \leqslant \vartheta$.

Proof Assume the construction breaks down at some j, i. e., h^j is defined, but h^{j+1} cannot be constructed. (Recall that if j is a limit ordinal and all h^i are constructed, it is a rather trivial matter to find h^j.) This can only happen in Case 2.2, whence j must be a limit ordinal. So either $\mu_{\alpha_j} > \widehat{\lambda}_j$ or $\alpha_j = \vartheta^{(\omega_1)}$.

Note first that $\widehat{\lambda}_j$ satisfies the normality requirement of the definition of μ_{α_j}: For assume that there existed some $\alpha < \alpha_j$ such that $\nu_\alpha \geqslant \widehat{\lambda}_j$. As j is a limit ordinal and $\alpha_j = \sup\{\alpha_i \mid i < j\}$, there must be some $i < j$ such that $\alpha_i > \alpha$ and thus $\mu_{\alpha_i} > \nu_\alpha$ (by the normality of $\mathcal{I}^M$). Pick i large enough such that $\lambda_j \in \mathrm{rge}(\pi^N_{ij})$, say $\lambda_j = \pi^N_{ij}(\bar{\lambda})$. By normality of $\mathcal{I}^N$, $\bar{\lambda} \geqslant \lambda_i$. But then

$$\begin{aligned}
\widehat{\lambda}_j = h^j(\pi^N_{ij}(\bar{\lambda})) &= \pi^M_{\alpha_i,\alpha_j}(h^i(\bar{\lambda})) \\
&\geqslant \pi^M_{\alpha_i,\alpha_j}(h^i(\lambda_i)) \\
&\geqslant \pi^M_{\alpha_i,\alpha_j}(\mu_{\alpha_i}) \\
&> \pi^M_{\alpha_i,\alpha_j}(\nu_\alpha) = \nu_\alpha \\
&\geqslant \widehat{\lambda}_j,
\end{aligned}$$

a contradiction! Thus the only reason why μ_{α_j} might be larger than $\widehat{\lambda}_j$ or not defined at all is that $\widetilde{m}^{\alpha_j}(\widehat{\lambda}_j) = \Lambda$.

By $(*)$ we know that then $\tilde{n}^j(\lambda_j) = \Lambda$, too. This will lead to a contradiction. We will find a sequence $\langle i_\xi \mid \xi < \Lambda \rangle$ of indices where n^i was increased at λ_j (or rather its pre-image). At each of these points, one extender with critical point (corresponding to) λ_j must have been used cofinally before. By sufficiently thinning out the sequence, we will show that at $i_{\omega_1^n}$ (where $n = o^{N^j}(\lambda_j)$), *all* available extenders must have been used up cofinally, leaving none to continue the iteration in a short way.

Fix some notation: for $i < j$ such that $\lambda_j \in \mathrm{rge}(\pi^N_{ij})$, let $\bar{\lambda}_i := (\pi^N_{ij})^{-1}(\lambda_j)$. For $\xi < \Lambda$, let $i_{\xi+1}$ be the least $i < j$, $i > i_\xi$, such that $n^i(\bar{\lambda}_i) = \tilde{n}^i(\bar{\lambda}_i) + 1$, and for limit ordinals $\zeta < \Lambda$, let $i_\zeta := \sup\{i_\xi \mid \xi < \zeta\}$. Such i must exists, as $\Lambda = \tilde{n}^j(\lambda_j) = \sup\{n^i(\bar{\lambda}_i) \mid i < j\}$. Note that then $\lambda_{i_{\xi+1}} = \bar{\lambda}_{i_{\xi+1}}$ and that $\mu_{\alpha_{i_{\xi+1}}} = h^{i_{\xi+1}}(\lambda_{i_{\xi+1}})$. By the subclaim of Case 2.1.2.3 of the construction of h^j, we know that

$$Q^N(i_{\xi+1}) = \{i' < i_{\xi+1} \mid \pi^N_{i' i_{\xi+1}}(\lambda_{i'}) = \lambda_{i_{\xi+1}}\} \text{ is unbounded in } i_{\xi+1}.$$

But it is also true that at the limit points i_ζ of the sequence the same critical point is used, i. e., $\lambda_{i_\zeta} = \bar{\lambda}_{i_\zeta}$: We cannot have $\lambda_{i_\zeta} > \bar{\lambda}_{i_\zeta}$, for then one would get the absurdity of

$$\lambda_j = \pi^N_{i_\zeta j}(\bar{\lambda}_{i_\zeta}) = \bar{\lambda}_{i_\zeta} < \lambda_{i_\zeta} < \lambda_j,$$

where the last inequality stems from the normality of the iteration $\mathcal{I}^N$. On the other hand, the normality also implies $\lambda_{i_\zeta} \geqslant \bar{\lambda}_{i_\zeta}$, for this critical point has been used cofinally up to i_ζ: Surely $\lambda_{i_\zeta} > \lambda_i$ for any $i < i_\zeta$. Now assume $\lambda_{i_\zeta} < \bar{\lambda}_{i_\zeta}$. Then, for some $i < i_\zeta$, $\lambda_{i_\zeta} = \pi^N_{i i_\zeta}(\tilde{\lambda}_i)$. Since $i_\zeta = \sup\{i_\xi \mid \xi < \zeta\}$, assume w.l.o.g. that $i = i_{\xi+1}$ for some such ξ. But then, keeping in mind that $\pi^N_{i i_\zeta}(\bar{\lambda}_i) = \bar{\lambda}_{i_\zeta}$ and $\bar{\lambda}_i = \lambda_i$, we must have $\tilde{\lambda}_i \geqslant \lambda_i$, as otherwise $\lambda_{i_\zeta} = \pi^N_{i i_\zeta}(\tilde{\lambda}_i) = \tilde{\lambda}_i < \lambda_i$, contradicting normality, and also $\tilde{\lambda}_i \leqslant \lambda_i$, as otherwise $\lambda_{i_\zeta} = \pi^N_{i i_\zeta}(\tilde{\lambda}_i) > \pi^N_{i i_\zeta}(\lambda_i) = \bar{\lambda}_{i_\zeta}$, which we excluded earlier on.

Thus we know that at every point of the sequence $\langle i_\xi \mid \xi < \Lambda\rangle$, $\lambda_{i_\xi} = \bar{\lambda}_{i_\xi}$, i. e., we use the same critical point at all these stages. Also, this same critical point has been used cofinally leading up to each i_ξ:

$$Q^N(i_\xi) \text{ is unbounded in } i_\xi$$

for *any* $\xi < \Lambda$ (and not only the successor ordinals).

We will now show that, setting $\Omega := \omega_1^n$, every possible *index* $k < n := o^{N^j}(\lambda_j)$ will be used cofinally leading up to i_Ω. But then we will arive at a contradiction: at stage i_Ω, we must again use some extender with the right critical point. But no matter which index is used, the iteration cannot have been short any more, contradicting the assumption at the outset that $\mathcal{I}^N$, indeed, was short.

To this end we will define sets and sequences of sets as well as (finite) ordinals and sequences of finite ordinals. The idea is that at any stage of the iteration from a prescribed set we can tell which of the (finitely many) extenders has been used. We consider Ω as a limit of ordinals of cofinality ω_1 such that each of the ordinals is again a limit ordinal of cofinality ω_1 such that etc., n-levels deep. In a first step, we choose one extender (i. e., an index $k^1 < n$) which will be used cofinally up to i_Ω at the limit points of the first level. Then, for each of these points leading up to i_Ω on the first level, we look at the limit points of the second level. Again, one extender must have been used cofinally, depending only on the point of the first level we are leading up to. By thinning out the first level now, we can assure that all these "second-level" extenders are, in fact, the same. Thus, after the second step of our construction, we will have shown that already two different extenders must have been used cofinally up to i_Ω, provided of course that the iteration $\mathcal{I}^N$ is short. Repeating this n-times shows that, in fact, *all* n-many extenders with critical point λ_{i_Ω} must have been used cofinally up to i_Ω, leaving none to continue $\mathcal{I}^N$ in a short way.

For the first step, in which A^1_1 and k^1 are defined, note that

$$\omega_1^n = \lim_{\zeta_1 \to \omega_1} \omega_1^{n-1}(\zeta_1 + 1),$$

and since the sequence $\langle i_\xi \mid \xi < \Lambda \rangle$ is a normal sequence, also

$$i_{\omega_1^n} = \lim_{\zeta_1 \to \omega_1} i_{\omega_1^{n-1}(\zeta_1+1)}.$$

To facilitate notation, let $e_\xi := \eta_{i_\xi}$, the index of the extender used at stage i_ξ of the iteration $\mathcal{I}^N$. Now there are only n possible values of e_ξ, so there must exist some $k^1 < n$ such that the extender with index k^1 has been used stationarily often, i. e., the set

$$A^1_1 := \{\zeta_1 < \omega_1 \mid e_{\omega_1^{n-1}(\zeta_1+1)} = k^1\}$$

is stationary in ω_1. Note that then

$$P^N(i_\Omega, k^1) = \{i < i_\Omega \mid \pi^N_{ii_\Omega}(\eta_i) = k^1\}$$

is unbounded in i_Ω, as it contains $\{i_{\omega_1^{n-1}(\zeta_1+1)} \mid \zeta_1 \in A^1_1\}$.

For the second step, first fix some $\zeta_1 \in A^1_1$. Now consider $\omega_1^{n-1}\zeta_1 + \omega_1^{n-1} = \lim_{\zeta_2 \to \omega_1}\left(\omega_1^{n-1}\zeta_1 + \omega_1^{n-2}(\zeta_2+1)\right)$. Thus,

$$i_{\omega_1^{n-1}(\zeta_1+1)} = \lim_{\zeta_2 \to \omega_1} i_{\omega_1^{n-1}\zeta_1 + \omega_1^{n-2}(\zeta_2+1)}.$$

As before, there must now exist some $k^2_1(\zeta_1)$ such that the set

$$A^2_2(\zeta_1) := \{\zeta_2 < \omega_1 \mid e_{\omega_1^{n-1}\zeta_1 + \omega_1^{n-2}(\zeta_2+1)} = k^2_1(\zeta_1)\}$$

is stationary in ω_1. Again, one concludes that

$$P^N(i_{\omega_1^{n-1}(\zeta_1+1)}, k^2(\zeta_1)) \text{ is unbounded in } i_{\omega_1^{n-1}(\zeta_1+1)}.$$

The iteration $\mathcal{I}^N$ was assumed to be short, so that $P^N(i_{\omega_1^{n-1}(\zeta_1+1)}, e_{\omega_1^{n-1}(\zeta_1+1)})$ must be *bounded* in $i_{\omega_1^{n-1}(\zeta_1+1)}$. And since ζ_1 was chosen from A^1_1, one has $e_{\omega_1^{n-1}(\zeta_1+1)} = k^1$. Thus it follows that $k^2_1(\zeta_1) \neq k^1$.

Next, notice that there must be some k^2 such that the set

$$A^1_2 := \{\zeta_1 \in A^1_1 \mid k^2_1(\zeta_1) = k^2\}$$

is stationary in ω_1. Obviously, $k^2 \neq k^1$. Furthermore,

$$P^N(i_\Omega, k^2) \text{ is unbounded in } i_\Omega,$$

as this set contains $\{i_{\omega_1^{n-1}\zeta_1 + \omega_1^{n-2}(\zeta_2+1)} \mid \zeta_1 \in A^1_1 \wedge \zeta_2 \in A^2_1(\zeta_1)\}$ and this latter set is unbounded in i_Ω.

Before writing down the induction step in full generality, we present the case $n = 3$ in some detail, believing that this will much better serve the purpose of shedding some light onto this rather unwieldy profusion of indices.

Let $q^1(\zeta_1) := \omega_1^{n-1}\zeta_1$ and $q^2(\zeta_1, \zeta_2) := \omega_1^{n-1}\zeta_1 + \omega_1^{n-2}\zeta_2$. Fix some $\zeta_1 \in A^1_2$, and some $\zeta_2 \in A^2_2(\zeta_1)$. Then the second step ensured that

$$e_{q^1(\zeta_1+1)} = k^1 \qquad \text{and} \qquad e_{q^2(\zeta_1,\zeta_2+1)} = k^2_1(\zeta_1) = k^2.$$

Set $q^3(\zeta_1,\zeta_2,\zeta_3) := \omega_1^{n-1}\zeta_1 + \omega_1^{n-2}\zeta_2 + \omega_1^{n-3}\zeta_3$ and note that

$$q^2(\zeta_1,\zeta_2+1) = \lim_{\zeta_3\to\omega_1} q^3(\zeta_1,\zeta_2,\zeta_3+1),$$

so that, since $\langle i_\xi \mid \xi < \Lambda\rangle$ is a normal sequence, also

$$i_{q^2(\zeta_1,\zeta_2+1)} = \lim_{\zeta_3\to\omega_1} i_{q^3(\zeta_1,\zeta_2,\zeta_3+1)}.$$

Thus there must be some $k_2^3(\zeta_1,\zeta_2) < n$ such that the set

$$A_3^3(\zeta_1,\zeta_2) := \{\zeta_3 < \omega_1 \mid e_{q^3(\zeta_1,\zeta_2,\zeta_3+1)} = k_2^3(\zeta_1,\zeta_2)\}$$

is stationary in ω_1. Thus, as before,

$$P^N(i_{q^2(\zeta_1,\zeta_2+1)}, k_2^3(\zeta_1,\zeta_2)) \text{ is unbounded in } i_{q^2(\zeta_1,\zeta_2+1)},$$

which implies, together with the shortness of $\mathcal{I}^N$, that

$$k_2^3(\zeta_1,\zeta_2) \neq k_1^2(\zeta_1) = k^2.$$

Now let ζ_2 range over elements of $A_2^2(\zeta_1)$. Surely there must exist some $k_1^3(\zeta_1)$ such that

$$A_3^2(\zeta_1) := \{\zeta_2 \in A_2^2(\zeta_1) \mid k_2^3(\zeta_1,\zeta_2) = k_1^3(\zeta_1)\}$$

is a stationary subset of $A_2^2(\zeta_1)$. Note that for a fixed ζ_1, the set

$$\{i_{q^3(\zeta_1,\zeta_2,\zeta_3+1)} \mid \zeta_2 \in A_3^2(\zeta_1) \ \wedge\ \zeta_3 \in A_3^3(\zeta_1,\zeta_2)\}$$

is unbounded in $i_{q^1(\zeta_1+1)}$, so that

$$P^N(i_{q^1(\zeta_1+1)}, k_1^3(\zeta_1)) \text{ is unbounded in } i_{q^1(\zeta_1+1)},$$

whence by the shortness of $\mathcal{I}^N$

$$k_1^3(\zeta_1) \neq e_{q^1(\zeta_1+1)} = k^1.$$

(The last equation holds as ζ_1 was chosen from $A_2^1 \subseteq A_1^1$.) Finally (for the third step), we need to thin out the set A_2^1 to get one value of k_1^3 working for all ζ_1: find some $k^3 < n$ such that the set

$$A_3^1 := \{\zeta_1 \in A_2^1 \mid k_1^3(\zeta_1) = k^3\}$$

$$
\begin{tikzcd}
A_1^1 \arrow[r, phantom, "\supseteq"] \arrow[dr, dashed] & A_2^1 \arrow[r, phantom, "\supseteq"] \arrow[ddr, dashed] & A_3^1 \arrow[r, phantom, "\supseteq"] \arrow[dddr, dashed] & A_4^1 & \cdots \arrow[r, phantom, "\supseteq"] & A_n^1 \\
 & A_2^2(\zeta_1) \arrow[r, phantom, "\supseteq"] \arrow[u, dashed] & A_3^2(\zeta_1) \arrow[r, phantom, "\supseteq"] \arrow[u, dashed] & A_4^2(\zeta_1) \arrow[u, dashed] & \cdots \arrow[r, phantom, "\supseteq"] & A_n^2(\zeta_1) \arrow[u, dashed] \\
 & & A_3^3(\zeta_1,\zeta_2) \arrow[r, phantom, "\supseteq"] \arrow[u, dashed] & A_4^3(\zeta_1,\zeta_2) \arrow[u, dashed] & \cdots \arrow[r, phantom, "\supseteq"] & A_n^3(\zeta_1,\zeta_2) \arrow[u, dashed] \\
 & & & \vdots \arrow[u, dashed] & & \vdots \arrow[u, dashed] \\
 & & & & & A_n^n(\zeta_1,\dots,\zeta_{n-1}) \arrow[u, dashed]
\end{tikzcd}
$$

Figure 6.3: Arrows indicate the order in which the sets are chosen in the construction. Sets in the l-th row contain "good" values for ζ_l. Sets in the m-th column represent combinations $\zeta_1,\dots,\zeta_m$ which simultaneously "exhaust" the extenders with indices from $\{k^1,\dots,k^m\}$.

is stationary in A_2^1. Obviously, we must have $k^3 \in \setminus\{k^1,k^2\}$. Furthermore,

$$\{i_{q^3(\zeta_1,\zeta_2,\zeta_3+1)} \mid \zeta_1 \in A_3^1 \ \wedge\ \zeta_2 \in A_3^2(\zeta_1) \ \wedge\ \zeta_3 \in A_3^3(\zeta_1,\zeta_2)\}$$

is unbounded in i_Ω, and at any such stage $e_{q^3(\zeta_1,\zeta_2,\zeta_3+1)} = k^3$, so that

$$P^N(i_\Omega,k^3) \text{ is unbounded in } i_\Omega.$$

Let $l \leqslant n$, and assume that all relevant (sequences of) sets and ordinals had been chosen (cf. Figure 6.3). Define $q^l(\zeta_1,\dots,\zeta_l) := \Sigma_{m=1}^l \omega_1^{n-m}\zeta_m$. Fix $\zeta_1 \in A_{l-1}^1$, $\zeta_2 \in A_{l-1}^2(\zeta_1)$, $\dots$, $\zeta_{l-1} \in A_{l-1}^{l-1}(\zeta_1,\dots,\zeta_{l-2})$. Note that

$$i_{q^{l-1}(\zeta_1,\dots,\zeta_{l-2},\zeta_{l-1}+1)} = \lim_{\zeta_l\to\omega_1} i_{q^l(\zeta_1,\dots,\zeta_{l-1},\zeta_l+1)}.$$

Thus there is some $k_{l-1}^l(\zeta_1,\dots,\zeta_{l-1}) < n$ such that

$$A_l^l(\zeta_1,\dots,\zeta_{l-1}) := \{\zeta_l < \omega_1 \mid e_{q^l(\zeta_1,\dots,\zeta_{l-1},\zeta_l+1)} = k_{l-1}^l(\zeta_1,\dots,\zeta_{l-1})\}$$

is stationary in ω_1. This also implies that

$P^N(i_{q^{l-1}(\zeta_1,\dots,\zeta_{l-2},\zeta_{l-1}+1)}, k_{l-1}^l(\zeta_1,\dots,\zeta_{l-1}))$ is unbounded in $i_{q^{l-1}(\zeta_1,\dots,\zeta_{l-2},\zeta_{l-1}+1)}$, whence by the shortness of $\mathcal{I}^N$ and the fact that $e_{q^{l-1}(\zeta_1,\ldots,\zeta_{l-2},\zeta_{l-1}+1)} = k^{l-1}$ it follows that

$$k^l_{l-1}(\zeta_1,\ldots,\zeta_{l-1}) \neq k^{l-1}.$$

Assume that $m < l$, $k^l_m(\zeta_1,\ldots,\zeta_m) < n$ is unequal to k^m, and

$$A^{m+1}_l(\zeta_1,\ldots,\zeta_m) = \\ \{\zeta_{m+1} \in A^{m+1}_{l-1}(\zeta_1,\ldots,\zeta_m) \mid k^l_{m+1}(\zeta_1,\ldots,\zeta_m,\zeta_{m+1}) = k^l_m(\zeta_1,\ldots,\zeta_m)\}$$

(if $m < l-1$, or

$$A^l_l(\zeta_1,\ldots,\zeta_{l-1}) = \{\zeta_l \in \omega_1 \mid e_{q^l(\zeta_1,\ldots,\zeta_{l-1},\zeta_l+1)} = k^l_{l-1}(\zeta_1,\ldots,\zeta_{l-1})\}$$

if $m = l-1$) is a stationary subset of $A^{m+1}_{l-1}(\zeta_1,\ldots,\zeta_m)$ (if $m < l-1$, or ω_1, if $m = l-1$). Then there must be some $k^l_{m-1} < n$ such that

$$A^m_l(\zeta_1,\ldots,\zeta_{m-1}) := \\ \{\zeta_m \in A^m_{l-1}(\zeta_1,\ldots,\zeta_{m-1}) \mid k^l_m(\zeta_1,\ldots,\zeta_m) = k^l_{m-1}(\zeta_1,\ldots,\zeta_{m-1})\}$$

is a stationary subset of $A^m_{l-1}(\zeta_1,\ldots,\zeta_{m-1})$. Again one concludes that $k^l_{m-1}(\zeta_1, \ldots,\zeta_{m-1})$ cannot be equal to $e_{q^{m-1}(\zeta_1,\ldots,\zeta_{m-1}+1)} = k^{m-1}$. Repeating this procedure l-times finally gives some $k^l \in n \setminus \{k^m \mid m < l\}$ such that for any $\zeta_1 \in A^1_l$, $\zeta_2 \in A^2_l(\zeta_1)$, $\ldots$, $\zeta_l \in A^l_l(\zeta_1,\ldots,\zeta_{l-1})$,

$$e_{q^l(\zeta_1,\ldots,\zeta_{l-1},\zeta_l+1)} = k^l,$$

and consequently

$$P^N(i_\Omega, k^l) \text{ is unbounded in } i_\Omega.$$

At the end of this procedure, one will have chosen n-many k^l, so that $\{k^l \mid l < n\} = n$. By the pigeon hole principle, we must have that $e_\Omega = k^l$, for some $l < n$. But then, at stage i_Ω, the iteration $\mathcal{I}^N$ cannot have been short. Contradiction. Thus the construction of the h^j cannot have broken down at some $j < \vartheta$ and we can indeed embed N^ϑ into $M^{\vartheta(\omega_1)}$. □(Subclaim)

□(Claim 3)

■

Our original goal was to prove in Chapter 7 that the existence of an irregular ultrafilter on κ implied the existence of an inner model for a strong cardinal. Currently, the restrictions on Lemma 6.2 prevent this. This Lemma is central to the proof of Claim 8 in the proof of Theorem 7.5. The following example shows that it cannot be extended straightforwardly to accomodate inner models for, say, strong cardinals.

Let M be a mouse of size ϱ, where $\kappa = \varrho^+$, with a measurable cardinal μ with $o^M(\mu) = \varrho$. Then for any $\zeta < \kappa$ there are iterations of M of length ζ in which any extender is used at most once (not even $<\omega_1$-many times). Just consider a well-ordering r of ϱ of ordertype ζ (assuming w.l.o.g. that $\zeta \geqslant \varrho$), and use the (image of the) $r(\xi)$-th extender on μ to construct the ξ-th step of the iteration. Hence the bound of the ordinal height of short iterates of M cannot be less than κ.

Chapter 7

Irregular Ultrafilters

If there wasn't anything to find out, it would be dull. Even trying to find out and not finding out is just as interesting as trying to find out and finding out; and I don't know but more so.

(Mark Twain, Eve's Diary)

Recall some basic ultrafilter definitions.

7.1 Definition Let $\mathcal{V}$ be an ultrafilter on some cardinal κ. Then $\mathcal{V}$ is called *uniform* iff every element of $\mathcal{V}$ has cardinality κ. $\mathcal{V}$ is called *normal* iff it is closed under diagonal intersections iff every regressive function (mod $\mathcal{V}$) is constant (mod $\mathcal{V}$). $\mathcal{V}$ is called *weakly normal* iff every regressive function (mod $\mathcal{V}$) is bounded below κ (mod $\mathcal{V}$).

From now on, $\mathcal{V}$ will always denote an ultrafilter on some cardinal κ.

7.2 Definition An ultrafilter $\mathcal{V}$ is *regular* iff there is a sequence $\langle a_\xi \mid \xi < \kappa \rangle$ of elements of $\mathcal{V}$ such that the intersection of any ω-many of them is empty. Such a sequence is then called *regularity sequence*.

More generally, $\mathcal{V}$ is (γ, κ)*-regular* (for some $\gamma < \kappa$) iff the intersection of any γ-many of them is empty. $\mathcal{V}$ is (γ, τ)-regular if the regularity sequence has length τ instead of κ. (This last concept will not be considered here.)

Finally, for the purpose of this thesis, call an ultrafilter $\mathcal{V}$ *weakly* (γ, κ)-*regular* iff the intersection of any γ-many elements of the sequence $\langle a_\xi \mid \xi < \kappa\rangle$ is bounded in κ.

So regularity is a (strong) form of incompleteness. Trivially, if $\mathcal{V}$ is (γ, κ)-regular, then it is (γ', κ)-regular for any $\gamma' > \gamma$. Recall that an ultrafilter $\mathcal{V}$ is γ-complete iff the intersection of any β-many sets from $\mathcal{V}$, $\beta < \gamma$, is again an element of $\mathcal{V}$.

Note the following (easy) equivalence, illustrated in Figure 7.1: $\mathcal{V}$ is (γ, κ)-regular iff there exists a (γ, κ)-*covering sequence* for $\mathcal{V}$, i. e., a family $\langle x_\eta \mid \eta < \kappa\rangle$ of subsets of κ, each of cardinality less than γ, such that for all $\xi < \kappa$, $\{\eta < \kappa \mid \xi \in x_\eta\} \in \mathcal{V}$.

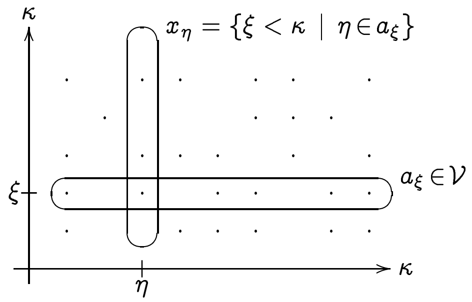

Figure 7.1: The two versions of (γ, κ)-regularity are just two ways of looking at the same diagramme. "Horizontal" sets a_ξ are elements from the regularity sequence, i. e., elements of $\mathcal{V}$, whereas "vertical" sets are elements from the covering sequence, i. e., of cardinality less than γ.

We are interested in the connection between weak regularity and regularity.

7.3 Lemma *Let $\varrho \geqslant \omega_1$ be a cardinal such that $\varrho^{\aleph_0} = \varrho$. Let $\kappa = \varrho^+$. Assume $\mathcal{V}$ is a uniform ultrafilter on κ which is weakly (ω, κ)-regular. Then $\mathcal{V}$ is (ω_1, κ)-regular.*

Proof Let $\mathcal{V}$ be a uniform ultrafilter on $\kappa > \omega_1$ which is weakly (ω, κ)-regular, and let $\langle a_\xi \mid \xi < \kappa\rangle$ witness this fact. We have to construct an

(ω_1, κ)-regularity sequence. This is done by induction on $\xi < \kappa$. The idea is to successively cut off the a_ζ so that they have empty intersection with any ω-many a_ξ preceeding them in the sequence. The assumption that $\varrho^{\aleph_0} = \varrho$ thus seems to be essential to the proof.

For $\xi < \omega$, set $b_\xi := a_\xi \in \mathcal{V}$. Now assume $\langle b_\xi \mid \xi < \zeta \rangle$ had been constructed, for some $\zeta < \kappa$. For any sequence $x = \langle \xi_i \mid i \in \omega \rangle$ of ordinals less than ζ, we know that $\bigcap_{i \in \omega} a_{\xi_i}$ is bounded in κ, since the sequence $\langle a_\xi \mid \xi < \kappa \rangle$ witnesses weak (ω, κ)-regularity. So let

$$\alpha_x := \sup \bigcap_{i \in \omega} a_{\xi_i} < \kappa$$

and let

$$\alpha_\zeta := \sup\{\alpha_x + 1 \mid x \in {}^\omega\zeta\}.$$

Since, by assumption, $\operatorname{card}(\zeta) \leqslant \varrho$ and $\varrho^{\aleph_0} = \varrho < \kappa$ it follows that $\alpha_\zeta < \kappa$. Set $b_\zeta := a_\zeta \setminus \alpha_\zeta$. $b_\zeta \in \mathcal{V}$, as $\mathcal{V}$ is uniform.

It remains to show that $\langle b_\xi \mid \xi < \kappa \rangle$ is an (ω_1, κ)-regularity sequence. So let $x = \langle \xi_i \mid i \in \omega_1 \rangle$ be a sequence of ordinals less than κ. By construction, $b_{\xi_i} \subseteq a_{\xi_i}$, and thus, considering the intersection of the first ω-many of these sets,

$$\bigcap_{i \in \omega} b_{\xi_i} \subseteq \bigcap_{i \in \omega} a_{\xi_i} \subseteq \alpha_{\langle \xi_i \mid i \in \omega \rangle} =: \bar{\alpha}.$$

But $\langle \xi_i \mid i \in \omega \rangle \in {}^\omega\xi_\omega$, so by the definition of α_{ξ_ω}, $\bar{\alpha} < \alpha_{\xi_\omega}$ and so

$$\Big(\bigcap_{i \in \omega} b_{\xi_i}\Big) \cap b_{\xi_\omega} \;\subseteq\; \bar{\alpha} \cap (\kappa \setminus \alpha_{\xi_\omega}) = \emptyset.$$

A fortiori, $\bigcap_{i \in \omega_1} b_{\xi_i} = \emptyset$, and $\langle b_\xi \mid \xi < \kappa \rangle$ is an (ω_1, κ)-regularity sequence. In fact, the proof shows that already the intersection of $(\omega + 1)$-many sets from the sequence is empty. ■

7.4 Definition An ultrafilter $\mathcal{V}$ is *fully irregular* iff it is not (γ, κ)-regular for any $\gamma < \kappa$.

7.5 Theorem *Let $\kappa = \varrho^+$ be successor cardinal, $\kappa \geqslant \omega_3$. Assume $2^\varrho = \varrho^+$ and $\varrho^{\aleph_0} = \varrho$. Let $\mathcal{V}$ be a fully irregular ultrafilter on κ. Then there is an inner model N satisfying $o^N(\mu) = \omega$ for some μ.*

Proof Assume for a contradiction that there is no such inner model. As we have mentioned before, this proof is written up so as to go through for the assumption that there was no inner model for a strong cardinal. It is only Lemma 6.2, which in turn is used in the proof of Claim 8 below, that forces us to make the much stronger assumption that there is no inner model satisfying $o^N(\mu) = \omega$. A proof based completely on this latter assumption would, of course, permit numerous simplifications, e. g., in the discussion of the various subcases, some of which are outright impossible under the stronger assumption.

Note that by [BK74, Corollary 2.2], cited as [Ket76, Theorem 1.11], we can conclude that $2^\kappa = \kappa^+$.

By a theorem of Kanamori and Ketonen ([Kan76], [Ket76]), we may assume that $\mathcal{V}$ is weakly normal. By [Kan76, Theorem 2.3], we may also assume that $\{\alpha \in \kappa \mid \mathrm{cf}(\alpha) \geqslant \omega_2\} \in \mathcal{V}$. Thus, if $x \subseteq \kappa$ is ω_2-club, then $x \in \mathcal{V}$.

Let $h : \kappa^+ \to \mathrm{H}_{\kappa^+}$ be a bijection. For $\tau \in (\kappa, \kappa^+)$ let f_τ be a surjection from κ onto τ, and let

$$F := \{\langle \xi, \zeta, \tau \rangle \mid f_\tau(\xi) = \zeta\}.$$

Write $F \restriction \tau$ for $F \cap (\kappa \times \tau \times \tau)$.

Let $\mathcal{H} = \langle \mathrm{H}_{\kappa^+}, \in, h, F \rangle$, $W := \mathrm{K}^c \downarrow \kappa^+$. For $\tau \in (\kappa, \kappa^+)$, let $\widetilde{\mathcal{H}}_\tau := \mathcal{H} \restriction (h''\tau)$, $H_\tau := h''\tau = |\widetilde{\mathcal{H}}_\tau|$, $W_\tau := H_\tau \cap W$. Let

$$I := \{\tau \in (\kappa, \kappa^+) \mid \mathrm{cf}(\tau) = \omega_2 \ \wedge\ \widetilde{\mathcal{H}}_\tau \prec \mathcal{H} \ \wedge\ \mathrm{On} \cap H_\tau = \tau \ \wedge\ \mathrm{H}_\kappa \subseteq H_\tau \\ \wedge\ H_\tau \text{ transitive} \ \wedge\ {}^{\omega_1} H_\tau \subseteq H_\tau \ \wedge\ W_\tau = W \downarrow \tau\}$$

7.6 Lemma *I is unbounded in κ.*

Proof The usual construction of building a tower of substructures of height ω_2, where each successive structure is the closure of the previous one under all relevant operations, yields the result. Note that $\mathrm{card}(\mathrm{H}_\kappa) = 2^\varrho = \kappa$, so that the initial step requiring $\mathrm{H}_\kappa \subseteq H$ is feasible. ■

For each $\tau \in I$, let $\langle \eta^\tau_\xi \mid \xi < \omega_2 \rangle$ be a cofinal sequence in τ. Let $\bar\eta^\tau_\xi := f_\tau^{-1}(h^{-1}(\eta^\tau_\xi)) < \kappa$. Since κ is regular, $\kappa > \omega_2$, we will have $\bar\eta^\tau := \sup\{\bar\eta^\tau_\xi \mid \xi < \omega_2\} < \kappa$.

For $\tau\in I$ let $\mathcal{H}_\tau := \langle H_\tau, \in, h\restriction\tau, F\restriction\tau, f_\tau\rangle$. For $\alpha\in\kappa$ let $\widetilde{\mathcal{H}}^\tau_\alpha := \mathcal{H}_\tau\restriction(h\circ f_\tau{}''\alpha)$ and $\widetilde{H}^\tau_\alpha := |\widetilde{\mathcal{H}}^\tau_\alpha| = h\circ f_\tau{}''\alpha$. Let

$$C_\tau := \{\alpha\in(\varrho,\kappa) \mid \widetilde{\mathcal{H}}^\tau_\alpha \prec \mathcal{H}_\tau \ \wedge\ \kappa\cap\widetilde{H}^\tau_\alpha = \alpha \ \wedge\ {}^{\omega_1}\widetilde{H}^\tau_\alpha \subseteq \widetilde{H}^\tau_\alpha \ \wedge\ \alpha > \bar{\eta}^\tau\}$$

and for $\alpha\in C_\tau$

$$\begin{array}{ll} \tilde{\pi}^\tau_\alpha : \mathcal{H}^\tau_\alpha \xrightarrow{\sim} \widetilde{\mathcal{H}}^\tau_\alpha, \quad \mathcal{H}^\tau_\alpha \text{ transitive} & \\ H^\tau_\alpha := |\mathcal{H}^\tau_\alpha|, & \tau_\alpha := \mathrm{On}\cap H^\tau_\alpha \\ K^\tau_\alpha := (\mathrm{K}^c)^{\mathcal{H}^\tau_\alpha}, & \pi^\tau_\alpha := \tilde{\pi}^\tau_\alpha\restriction K^\tau_\alpha \end{array}$$

Since $\alpha = \mathrm{crit}(\pi^\tau_\alpha) > \bar{\eta}^\tau$, we have $f_\tau{}''\bar{\eta}^\tau \subseteq \widetilde{H}^\tau_\alpha$, so $\mathrm{On}\cap\widetilde{H}^\tau_\alpha$ is cofinal in τ. This implies $\pi^\tau_\alpha : K^\tau_\alpha \to_e W_\tau$ cofinally.

7.7 Lemma *i) For $\tau\in I$, C_τ is ω_2-club, hence $C_\tau\in\mathcal{V}$.*
ii) ${}^{\omega_1}H^\tau_\alpha \subseteq H^\tau_\alpha$.
iii) $\pi^\tau_\alpha : K^\tau_\alpha \to_e W_\tau \prec W$
iv) $\mathrm{crit}(\pi^\tau_\alpha) = \alpha$.
v) $K^\tau_\alpha{\downarrow}\alpha = K{\downarrow}\alpha$.

Proof This is straightforward. ■

7.8 Lemma *Let $\sigma,\tau\in I$, $\tau\in\mathrm{rge}(\pi^\sigma_\alpha)$, $\alpha\in C_\tau\cap C_\sigma$. Then*

$$\begin{array}{ll} \mathcal{H}^\tau_\alpha \prec \mathcal{H}^\sigma_\alpha & \tilde{\pi}^\tau_\alpha = \tilde{\pi}^\sigma_\alpha\restriction\mathcal{H}^\tau_\alpha \\ K^\tau_\alpha \prec K^\sigma_\alpha & \pi^\tau_\alpha = \pi^\sigma_\alpha\restriction K^\tau_\alpha \end{array}$$

and, in fact, $\mathcal{H}^\tau_\alpha = (\tilde{\pi}^\sigma_\alpha)^{-1}(\mathcal{H}_\tau)\in\mathcal{H}^\sigma_\alpha$ and $K^\tau_\alpha = (\pi^\sigma_\alpha)^{-1}(W_\tau)$.

Proof Note that $\mathcal{H}_\tau$ is definable from τ in $\mathcal{H}_\sigma$, since $H_\tau = h''\tau = (h\restriction\mathcal{H}_\sigma)''\tau$ etc. Thus $\mathcal{H}_\tau \in \mathcal{H}_\sigma$. But then $\widetilde{\mathcal{H}}^\sigma_\alpha \cap \mathcal{H}_\tau \prec \mathcal{H}_\sigma$. Since every $\mathcal{H} \prec \mathcal{H}_\sigma$ is uniquely determined by $|\mathcal{H}|\cap\kappa$ via h, we get that $\widetilde{\mathcal{H}}^\tau_\alpha = \widetilde{\mathcal{H}}^\sigma_\alpha\cap\mathcal{H}_\tau$. But $\widetilde{\mathcal{H}}^\sigma_\alpha$ is an end-extension of $\widetilde{\mathcal{H}}^\sigma_\alpha\cap\mathcal{H}_\tau$, by which the rest follows. ■

The following lemma [DJK81, Lemma 2.3] (cf. also Lemma 4.4) will frequently be used.

7.9 LEMMA *Let $C \in \mathcal{V}$ and $f \in \prod_{\alpha\in C} \alpha^+$. Then there is a $\sigma \in I$ such that*

$$\{\alpha \in C \cap C_\sigma \mid f(\alpha) < \sigma_\alpha\} \in \mathcal{V}.$$

If $\{\alpha \in \kappa \mid \varphi(\alpha)\} \in \mathcal{V}$, say that $\varphi(\alpha)$ holds for $\mathcal{V}$-*almost all* α, or also $\forall^{\mathcal{V}}\alpha\, \big(\varphi(\alpha)\big)$.

CLAIM 1 $\forall \tau \in I\ \forall^{\mathcal{V}}\alpha \in C_\tau\ \big(K^\tau_\alpha$ is not weakly full$\big)$.

PROOF This proof is modelled after the proof of a similar claim in [Sch96, Lemma 5.6]. Pick some $\tau \in I$.

CASE 1 κ not overlapped in W.

In this case, we have that $\forall \mu < \kappa\ \big(O^W(\mu) < \kappa\big)$ and thus $E^W_\kappa = \emptyset$.

CASE 1.1 $\forall^{\mathcal{V}}\alpha \in C_\tau\ \big(E^W_\alpha = \emptyset\big)$.

Pick such an α. By elementarity of π^τ_α we have $E^{K^\tau_\alpha}_\alpha = \emptyset$, and also $E^W \restriction \alpha = E^{K^\tau_\alpha} \restriction \alpha$. It follows from 5.9 that $W \downarrow \min\{\alpha^{+W}, \alpha^{+K^\tau_\alpha}\} = K^\tau_\alpha \downarrow \min\{\alpha^{+W}, \alpha^{+K^\tau_\alpha}\}$. We claim that $\alpha^{+K^\tau_\alpha} \leqslant \alpha^{+W}$. Assume not. Then $\alpha^{+K^\tau_\alpha} > \alpha^{+W}$, and an initial segment of $K^\tau_\alpha \downarrow \alpha^{+K^\tau_\alpha}$ is a collapsing mouse for $W \downarrow \alpha^{+W}$. (Note that $\kappa^{W\downarrow\alpha^{+W}} = \alpha$.) Since W is weakly full, this initial segment would be a subset of W, clearly a contradiction.

If $\alpha^{+K^\tau_\alpha} < \alpha^{+W}$, we can similarly deduce that K^τ_α is not weakly full: an initial segment of $W \downarrow \alpha^{+W}$ is a collapsing mouse for $K^\tau_\alpha \downarrow \alpha^{+K^\tau_\alpha}$ which cannot be contained in K^τ_α.

So assume that $\alpha^{+K^\tau_\alpha} = \alpha^{+W}$. This will lead to a contradiction. We can conclude $\mathcal{P}(\alpha) \cap W = \mathcal{P}(\alpha) \cap K^\tau_\alpha$. As κ is not overlapped in W, $\nu := O^W(\alpha) < \kappa = \pi^\tau_\alpha(\alpha)$. Thus we can define an extender F at α, ν on W: For $a \in [\nu]^{<\omega}$, $x \in \mathcal{P}\left([\alpha]^{\overline{\overline{a}}}\right) \cap W$ set

$$x \in F_a \ \leftrightarrow\ a \in \pi^\tau_\alpha(x).$$

Since H^τ_α is ω-complete and $\mathcal{P}(\alpha) \cap W = \mathcal{P}(\alpha) \cap K^\tau_\alpha$, the usual argument shows that F is countably complete: Let $\langle x_n \mid n \in \omega\rangle$ be a sequence such that for $n \in \omega$, $x_n \in F_{a_n}$, where $a_n \in [\nu]^{<\omega}$. We need to show that there is a function $h : \bigcup_{n\in\omega} a_n \to \alpha$ such that for all $n \in \omega$, $h''a_n \in x_n$. Let

$\xi = \mathrm{otp}(\bigcup_{n\in\omega} a_n) < \omega_1$, and let $f : \xi \to \bigcup a_n$ be the monotone enumeration. Since $\mathcal{P}(\alpha) \cap W \subseteq K^\tau_\alpha$, we have $x_n \in K^\tau_\alpha$, too. And since $\widetilde{H}^\tau_\alpha$ is closed under ω-sequences, it follows that $\langle\langle x_n, f^{-1\prime\prime}a_n\rangle \mid n\in\omega\rangle \in \widetilde{H}^\tau_\alpha$. Notice that the function f is an element of H_τ: $\xi < \omega_1 < \kappa$ and also $\sup(\bigcup_{n\in\omega} a_n) < \kappa$, since $\nu < \kappa$, $a_n \subseteq \nu$, and $\mathrm{cf}(\kappa) = \kappa > \omega$. So $f \in \mathrm{H}_\kappa \subseteq H_\tau$. Also, $\nu < \kappa = \pi^\tau_\alpha(\alpha)$, and $a_n \in \pi^\tau_\alpha(x_n)$ by the definition of F, so

$$H_\tau \models \exists g\ \Big(g : \xi \to \pi^\tau_\alpha(\alpha)\ \wedge\ g \text{ is orderpreserving } \wedge \\ \forall n\in\omega\ \Big(g''(f^{-1\prime\prime}a_n) \in \pi^\tau_\alpha(x_n)\Big)\Big),$$

since f is a possible candidate. By elementarity, one concludes that

$$\widetilde{H}^\tau_\alpha \models \exists g\ \Big(g : \xi \to \alpha\ \wedge\ g \text{ is orderpreserving } \wedge\ \forall n\in\omega\ \Big(g''(f^{-1\prime\prime}a_n) \in x_n\Big)\Big).$$

Pick any such g and set $h := g\circ f^{-1}$. Then $h : \bigcup_{n\in\omega} a_n \to \alpha$ is orderpreserving and for $n\in\omega$, $h''a_n \in x_n$.

To reach a contradiction, it now suffices to show that if $j : W{\downarrow}\nu \to_F \mathrm{Ult}(W{\downarrow}\nu, F) =: \widetilde{W}$, then $\widetilde{W}|\nu = W{\downarrow}\nu$ and $\langle W{\downarrow}\nu, F\rangle$ is weakly amenable. For then, by the definition of W, one concludes that $E^W_\nu = F$, contradicting the fact that $\nu = O^W(\alpha)$. For what follows, refer to Figure 7.2. Since $W{\downarrow}\alpha^{+W} = K^\tau_\alpha{\downarrow}\alpha^{+K^\tau_\alpha}$, we have $\widetilde{W}{\downarrow}j(\alpha) = \mathrm{Ult}(K^\tau_\alpha{\downarrow}\alpha^{+K^\tau_\alpha}, F){\downarrow}j(\alpha)$. F was derived from π^τ_α, so there is a canonical map $k : \mathrm{Ult}(K^\tau_\alpha{\downarrow}\alpha^{+K^\tau_\alpha}, F) \to_{\Sigma_1} W{\downarrow}\kappa^{+W}$ such that $k\circ j \restriction (K^\tau_\alpha{\downarrow}\alpha^{+K^\tau_\alpha}) = \pi^\tau_\alpha \restriction (K^\tau_\alpha{\downarrow}\alpha^{+K^\tau_\alpha})$ and $\mathrm{crit}(k) \geqslant \nu$. But this now immediately implies that $\widetilde{W}|\nu = W|\nu = \langle W{\downarrow}\nu, \emptyset\rangle$ and also that $\langle W{\downarrow}\nu, F\rangle$ is weakly amenable, as desired. (Note that $\nu \geqslant \alpha^{+W}$, so that $\mathcal{P}(\alpha)\cap W = \mathcal{P}(\alpha)\cap W|\nu = \mathcal{P}(\alpha)\cap \widetilde{W}|\nu$.)

Case 1.2 $\forall^{\mathcal{V}}\alpha\in C_\tau\ \Big(E^W_\alpha \neq \emptyset\Big)$.

For such α, E^W_α is an extender at some μ_α, α on $W{\downarrow}\alpha$.

Case 1.2.1 $\forall^{\mathcal{V}}\alpha\in C_\tau\ \Big(\alpha < \mu^{+W}_\alpha\Big)$.

Notice that then $\alpha \notin \mathrm{Card}^W$ and $\kappa^{K^\tau_\alpha{\downarrow}\alpha} = \mu_\alpha$. Take γ_α minimal such that $\omega\varrho^\omega_{W\|\gamma_\alpha} \leqslant \mu_\alpha$, but $\mathcal{P}(\mu_\alpha)\cap W\|\gamma_\alpha \subseteq W{\downarrow}\alpha = K^\tau_\alpha{\downarrow}\alpha$. Then $W\|\gamma_\alpha$ is a collapsing mouse for $K^\tau_\alpha{\downarrow}\alpha$ ($\kappa^{K^\tau_\alpha{\downarrow}\alpha} = \mu_\alpha < \alpha$!). Notice $\gamma_\alpha < \mu^{+W}_\alpha \leqslant \kappa = \alpha^+$. We claim that for $\mathcal{V}$-almost all α, $W\|\gamma_\alpha \not\subseteq K^\tau_\alpha$. To see this, use Lemma 7.9 to find $\sigma\in I$ such that $\tau\in\mathrm{rge}(\pi^\sigma_\alpha)$ and $\gamma_\alpha < \sigma_\alpha$ for $\mathcal{V}$-almost all α. Then

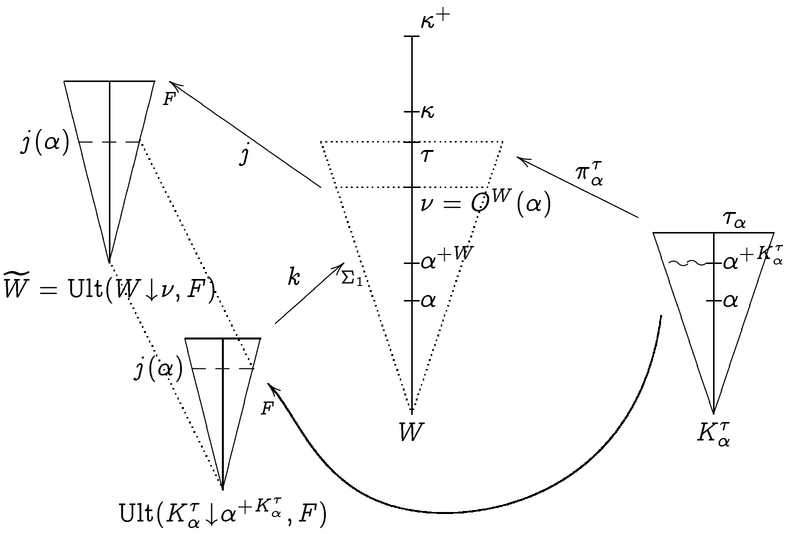

Figure 7.2: F is an extender at α, ν , so that $j(\alpha) \geqslant \nu$ and thus $\tilde{W}|\nu = W{\downarrow}\nu$.

we would have $K^\sigma_\alpha \models \alpha \notin \mathrm{Card}$, a contradiction. Thus in this case, too, we can conclude that K^τ_α is not weakly full (for $\mathcal{V}$-almost all α). Note that we need to go up to σ as the assumption $W\|\gamma_\alpha \subseteq K^\tau_\alpha$ does *not* already imply $\alpha \notin \mathrm{Card}^{K^\tau_\alpha}$.

Case 1.2.2 $\forall^{\mathcal{V}}\alpha \in C_\tau \left(\alpha \geqslant \mu_\alpha^{+W}\right)$.

Then $O^W(\mu_\alpha) > \alpha$. Note that $\pi^\tau_\alpha(\mu_\alpha) = \mu_\alpha$. If $\mu_\alpha^{+K^\tau_\alpha} \geqslant \alpha$, then

$$\mu_\alpha^{+W} = \pi^\tau_\alpha(\mu_\alpha)^{+K^\tau_\alpha} = \pi^\tau_\alpha(\mu_\alpha^{+K^\tau_\alpha}) \geqslant \pi^\tau_\alpha(\alpha) = \kappa,$$

implying the obvious contradiction $\alpha \geqslant \kappa$. Thus we must have $\mu_\alpha^{+K^\tau_\alpha} < \alpha$. Since $K^\tau_\alpha{\downarrow}\alpha = W{\downarrow}\alpha$, it follows that $O^{K^\tau_\alpha}(\mu_\alpha) \geqslant \alpha$. This in turn implies $O^W(\mu_\alpha) \geqslant \pi^\tau_\alpha(O^{K^\tau_\alpha}(\mu_\alpha)) \geqslant \kappa$, contradicting the assumption of Case 1 that κ is not overlapped in W.

Case 2 κ overlapped in W.

Say, $O^W(\mu) \geqslant \kappa$, for some $\mu < \kappa$.

Case 2.1 $\forall^{\mathcal{V}}\alpha \in C_\tau \left(\mathcal{P}(\alpha) \cap W \subseteq K^\tau_\alpha\right)$.

Pick some such α and define $\mathcal{U} := \{x \in \mathcal{P}(\alpha) \cap K^\tau_\alpha \mid \alpha \in \pi^\tau_\alpha(x)\}$. Let

$N := \langle K_\alpha^\tau{\downarrow}\alpha^{+K_\alpha^\tau}, \mathcal{U}\rangle$. Then

$$N \models \text{“}\mathcal{U} \text{ is a non-trivial, } \alpha\text{-complete ultrafilter on } \alpha\text{”}.$$

$\mathcal{U}$ is countably complete: Let $\langle x_n \mid n \in \omega\rangle$ be a sequence of elements of $\mathcal{U}$. As H_α^τ is ω-closed, $\langle x_n \mid n \in \omega\rangle \in H_\alpha^\tau$. By the definition of $\mathcal{U}$ one has that

$$\forall n \in \omega \ \Big(\alpha \in \pi_\alpha^\tau(x_n)\Big),$$

whence

$$\mathrm{H}_{\kappa^+} \models \exists\beta \ \forall n \in \omega \ \Big(\beta \in \pi_\alpha^\tau(x_n)\Big),$$

and by elementarity of π_α^τ

$$H_\alpha^\tau \models \exists\beta \ \forall n \in \omega \ \Big(\beta \in x_n\Big),$$

so that $\bigcap_{n\in\omega} x_n \neq \emptyset$. Furthermore, N is amenable: Let $x \in K_\alpha^\tau \downarrow \alpha^{+K_\alpha^\tau}$ be arbitrary. Pick some function $f \in K_\alpha^\tau$ mapping α onto $x \cap \mathcal{P}(\alpha) \cap K_\alpha^\tau$. Then

$$x \cap \mathcal{U} = \Big\{y \in \mathcal{P}(\alpha) \cap K_\alpha^\tau \mid y \in x \ \wedge \ \exists\xi < \alpha \ \Big(\alpha \in \pi_\alpha^\tau(f)(\xi)\Big)\Big\}.$$

Since $x \cap \mathcal{P}(\alpha) \cap K_\alpha^\tau \in W$, $x \cap \mathcal{U} \in W$. Thus $x \cap \mathcal{U} \in \mathcal{P}(\alpha) \cap W \subseteq K_\alpha^\tau$, and by acceptability of K_α^τ, $x \cap \mathcal{U} \in K_\alpha^\tau{\downarrow}\alpha^{+K_\alpha^\tau}$. Thus N is a neat p-premouse and $\mathrm{L}^{\text{strong}}$ exists by 5.14. This is a contradiction.

Case 2.2 $\forall^{\mathcal{V}}\alpha \in C_\tau \ \Big(\mathcal{P}(\alpha) \cap W \not\subseteq K_\alpha^\tau\Big)$.

Case 2.2.1 $\mu^{+W} = \kappa$.

In this case we get the required collapsing mice for K_α^τ fairly soon: By elementarity of π_α^τ, $\mu^{+K_\alpha^\tau} = \alpha < \kappa = \mu^{+W}$. So let γ_α be minimal such that $\omega\varrho^\omega_{W\|\gamma_\alpha} \leqslant \mu$, $\gamma_\alpha \geqslant \alpha$. Note that $\kappa^{K_\alpha^\tau\downarrow\alpha} = \mu$, and $K_\alpha^\tau \downarrow \alpha = W \downarrow \alpha$. By the minimality of γ_α, we have $\mathcal{P}(\mu) \cap W\|\gamma_\alpha \subseteq K_\alpha^\tau{\downarrow}\alpha$. Thus $W\|\gamma_\alpha$ is a collapsing mouse for $K_\alpha^\tau \downarrow \alpha$. If $W\|\gamma_\alpha$ were contained in K_α^τ then we would get a contradiction as in Case 1.2.1. Thus K_α^τ is not weakly full for $\mathcal{V}$-almost all α.

Case 2.2.2 $\mu^{+W} < \kappa$.

W.l.o.g. $\alpha > \mu^{+W}$, and so $\mu^{+K_\alpha^\tau} = \pi_\alpha^\tau(\mu^{+K_\alpha^\tau}) = \mu^{+W}$, $\mathcal{P}(\mu) \cap W = \mathcal{P}(\mu) \cap K_\alpha^\tau$, and $O^{K_\alpha^\tau}(\mu) \geqslant \alpha$.

CASE 2.2.2.1 $O^W(\mu) < \kappa^+$.

We can assume w.l.o.g. that for all $\tau \in I$, $O^W(\mu) < \tau$. Thus $\forall^{\mathcal{V}} \alpha \in C_\tau \left(O^{K^\tau_\alpha}(\mu) < \tau_\alpha\right)$. Pick τ and α, and let $\nu_\alpha := O^{K^\tau_\alpha}(\mu) < \tau_\alpha < \kappa \leqslant O^W(\mu)$. Then by the Condensation Lemma [Koe89, Theorem 22.3] we have $W{\downarrow}\nu_\alpha = K^\tau_\alpha|\nu_\alpha$.

As in Case 1.1, we aim to derive a contradiction from defining an extender F at μ, $\eta := O^W(\mu)$ from (more or less) π^τ_α. However, π^τ_α has critical point α and $\pi^\tau_\alpha(\alpha)$ is equal to κ, which in turn is less than or equal to η, so the direct approach from Case 1.1 will fail. We will thus first take an ultrapower of W with the extender at μ, $\nu_\alpha = O^{K^\tau_\alpha}(\mu)$, with embedding i, and then lift π^τ_α to this ultrapower, giving some map j. The composition k of i and j will then suffice to define an appropriate extender F (see Figure 7.3).

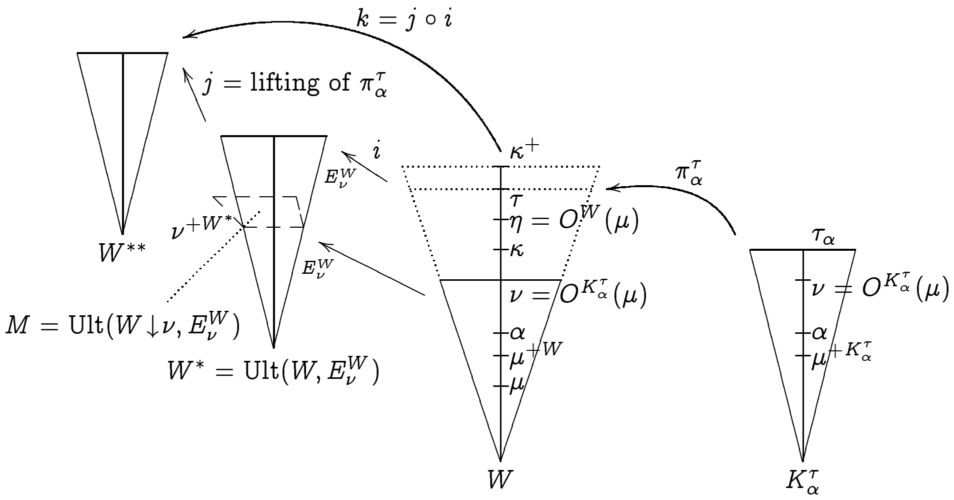

Figure 7.3: Let $i : W \to W^* = \mathrm{Ult}(W, E^W_{O^{K^\tau_\alpha}(\mu)})$. Lift π^τ_α to W^*, giving j, and let $k := j \circ i$. Then define an extender F at μ, η from k and go for a contradiction.

Let $M_\alpha := \mathrm{Ult}(W{\downarrow}\nu_\alpha, E^W_{\nu_\alpha})$. Then $M_\alpha|\nu_\alpha = W{\downarrow}\nu_\alpha = K^\tau_\alpha|\nu_\alpha$, and $E^{M_\alpha}_{\nu_\alpha} = E^{K^\tau_\alpha}_{\nu_\alpha} = \emptyset$. Let $\eta_\alpha := \min\{\nu_\alpha^{+K^\tau_\alpha}, \nu_\alpha^{+M_\alpha}\}$. Then by Corollary 5.9

$$M_\alpha{\downarrow}\eta_\alpha = K^\tau_\alpha{\downarrow}\eta_\alpha.$$

If $\nu_\alpha^{+M_\alpha} > \nu_\alpha^{+K_\alpha^\tau}$, then an initial segment of M_α is a collapsing mouse omitted in K_α^τ, and K_α^τ is not weakly full, as claimed.

So assume $\nu_\alpha^{+M_\alpha} \leqslant \nu_\alpha^{+K_\alpha^\tau}$. Then, since $M_\alpha{\downarrow}\nu_\alpha^{+M_\alpha} = K_\alpha^\tau{\downarrow}\nu_\alpha^{+M_\alpha}$, $\mathcal{P}(\nu_\alpha) \cap M_\alpha \subseteq K_\alpha^\tau$. This will lead to a contradiction. For the sake of legibility, denote ν_α by ν, M_α by M etc. Let $i : W \to_{E_\nu^W} W^*$. Then $W^* \downarrow \nu = M \downarrow \nu$ and $E_\nu^{W^*} = E_\nu^M = \emptyset$. In fact, the two ultrapowers agree up to $\nu^{+M} = \nu^{+W^*}$, since ν is the index of the extender used, and $\mathcal{P}(\mu) \cap W{\downarrow}\nu = \mathcal{P}(\mu) \cap W$.

Since $\nu^{+M} \leqslant \nu^{+K_\alpha^\tau}$, we can thus construct the upward extension of $\pi_\alpha^\tau \restriction (K_\alpha^\tau{\downarrow}(\nu^{+M}))$ to W^*, say $j : W^* \to W^{**}$. Notice that $\nu^{+M} \in \mathrm{Card}^{W^*}$. Let $k := j \circ i$ denote the composition, $k : W \to W^{**}$. Let $\eta := O^W(\mu)$. Then $k(\mu) = j(i(\mu)) \geqslant j(\nu) = j(O^{K_\alpha^\tau}(\mu)) = \pi_\alpha^\tau(O^{K_\alpha^\tau}(\mu)) = O^W(\mu) = \eta$. So we can derive an extender F at μ, η on W by setting, for $a \in [\eta]^{<\omega}$ and $x \in \mathcal{P}\left([\mu]^{\overline{\overline{a}}}\right)$,

$$x \in F_a \ \leftrightarrow\ a \in k(x).$$

Noting that $\mathcal{P}(\mu) \cap W = \mathcal{P}(\mu) \cap K_\alpha^\tau$, one sees that the same argument as in Case 1.1 shows that F must be countably complete. In fact, $\langle W{\downarrow}\eta, F\rangle$ is weakly amenable, and if $h : W{\downarrow}\eta \to_F \widetilde{W}$ denotes the ultrapower map then $\widetilde{W}|\eta = W{\downarrow}\eta$ (see Figure 7.4).

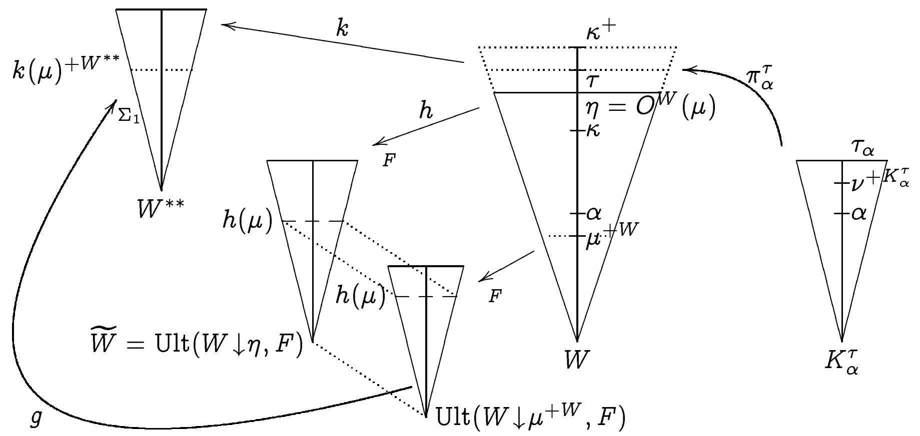

Figure 7.4: $\widetilde{W}|\eta = W{\downarrow}\eta$.

We have $\widetilde{W}{\downarrow}h(\mu) = \mathrm{Ult}(W{\downarrow}\mu^{+W}, F){\downarrow}h(\mu)$. Since F was derived from k, there is the canonical map $g : \mathrm{Ult}(W{\downarrow}\mu^{+W}, F) \to_{\Sigma_1} W^{**}{\downarrow}k(\mu)^{+W^{**}}$ such that $g \circ h \restriction (W{\downarrow}\mu^{+W}) = k$ and $\mathrm{crit}(g) \geqslant \eta$. Thus $\widetilde{W}{\downarrow}\eta = W^{**}{\downarrow}\eta$. Furthermore, $O^{W^{**}}(\mu) = j(O^{W^*}(\mu)) = j(\nu) = j(O^{K^\tau_\alpha}(\mu)) = \pi^\tau_\alpha(O^{K^\tau_\alpha}(\mu)) = O^W(\mu) = \eta$, so $E^{\widetilde{W}}_\eta = \emptyset$. Also, j was the upward extension of $\pi^\tau_\alpha \restriction (K^\tau_\alpha{\downarrow}\nu^{+K^\tau_\alpha})$, so W^{**} is an extension of $W{\downarrow}\sup(\pi^\tau_\alpha{}''\nu^{+K^\tau_\alpha})$. Since $\sup(\pi^\tau_\alpha{}''\nu^{+K^\tau_\alpha}) > \pi^\tau_\alpha(\nu) = O^W(\mu) = \eta$, we get $\widetilde{W}|\eta = W^{**}|\eta = W|\eta$. Since $\eta \geqslant \kappa > \mu^{+W}$, we conclude that $\mathcal{P}(\mu) \cap W \subseteq W{\downarrow}\eta = \widetilde{W}{\downarrow}\eta$, so $\langle W{\downarrow}\eta, F\rangle$ is indeed weakly amenable.

But taking all this together, one concludes that by the definition of W one should have $E^W_\eta = F$, whereas, by the choice of $\eta = O^W(\mu)$, $E^W_\eta = \emptyset$ — contradiction.

Case 2.2.2.2 $O^W(\mu) = \kappa^+$.

In this case, for all $\tau \in I$, $\forall^{\mathcal{V}}\alpha \in C_\tau \left(O^{K^\tau_\alpha}(\mu) = \tau_\alpha\right)$. By the Condensation Lemma [Koe89, Theorem 22.3] we have $K^\tau_\alpha = W{\downarrow}\tau_\alpha$. First we show κ to be inaccessible in W. Assume to the contrary that $\kappa = \lambda^{+W}$ and pick for each $\alpha \in C_\tau$ a surjection $h_\alpha : \lambda \to \alpha$, $h_\alpha \in W$. Then $h_\alpha \in W{\downarrow}\gamma_\alpha$ for some $\gamma_\alpha < \kappa = \alpha^+$. Now pick $\sigma \in I$ such that, for $\mathcal{V}$-almost all α, $\tau \in \mathrm{rge}(\pi^\sigma_\alpha)$ and $\gamma_\alpha < \sigma_\alpha$. Note that, since also $\forall^{\mathcal{V}}\alpha \in C_\sigma \left(O^{K^\sigma_\alpha}(\mu) = \sigma_\alpha\right)$, $K^\sigma_\alpha = W{\downarrow}\sigma_\alpha$, again by the Condensation Lemma. Then for such α we will have $K^\sigma_\alpha \models$ "α is not a cardinal", obviously a contradiction.

Hence κ is inaccessible in W. Thus for each $\alpha < \kappa$ there is $\gamma_\alpha < \kappa = \alpha^+$ such that $\mathcal{P}(\alpha) \cap W \in W{\downarrow}\gamma_\alpha$. Using Lemma 7.9, find $\sigma \in I$ such that, for $\mathcal{V}$-almost all α, $\tau \in \mathrm{rge}(\pi^\sigma_\alpha)$ and $\sigma_\alpha > \gamma_\alpha$. Since we have full condensation it follows that $\mathcal{P}(\alpha) \cap W \in K^\sigma_\alpha$. By elementarity, $\mathcal{P}(\alpha) \cap W \in K^\tau_\alpha$, too. But the assumption of Case 2.2 is that $\mathcal{P}(\alpha) \cap W \not\subseteq K^\tau_\alpha$, contradiction.

This proves the claim. $\square$(Claim 1)

Assume w.l.o.g. that Claim 1 holds for all $\tau \in I$ and all $\alpha \in C_\tau$. Note that since H^τ_α is closed under ω-sequences K^τ_α is neat and thus prestrong. If $\kappa^{K^\tau_\alpha} < \tau_\alpha$, then K^τ_α must have a collapsing mouse: otherwise it would be presolid, and since a presolid, prestrong mouse is solid, and since solid mice are weakly full, it would also be weakly full. Thus either $\kappa^{K^\tau_\alpha} = \tau_\alpha$ or K^τ_α has a collapsing mouse.

Case A First, let us analyse the case that there is some $\tau \in I$ such that for $\mathcal{V}$-almost all α we have $\kappa^{K^\tau_\alpha} < \tau_\alpha$ and hence K^τ_α has a collapsing mouse. By elementarity, we must have $\kappa^W < \kappa^+$ and so, in fact, this is true for all σ. Let N_α^τ denote the collapsing mouse for K_α^τ.

Denote $\kappa^{K_\alpha^\tau}$ by κ_α^τ. Then if $\alpha \in C_{\tau\sigma}$, it follows that $\kappa_\alpha^\tau = \kappa_\alpha^\sigma$ by elementarity.

Claim 2 $\forall \tau \in I \left(\kappa_\alpha^\tau \text{ is the largest cardinal in } K_\alpha^\tau\right)$.

Proof We first show that by increasing τ, one can always increase the power set of κ_α^τ:

$$\forall \tau \in I\ \exists \sigma \in I\ \left(\{\alpha \in C_{\tau\sigma} \mid \mathcal{P}(\kappa_\alpha^\tau) \cap K_\alpha^\tau \subsetneq \mathcal{P}(\kappa_\alpha^\sigma) \cap K_\alpha^\sigma\} \in \mathcal{V}\right).$$

Assume to the contrary that τ is such that

$$\forall \sigma \in I,\ \sigma > \tau\ \left(\{\alpha \in C_{\tau\sigma} \mid \mathcal{P}(\kappa_\alpha^\tau) \cap K_\alpha^\tau = \mathcal{P}(\kappa_\alpha^\sigma) \cap K_\alpha^\sigma\} \in \mathcal{V}\right).$$

But then for any $\sigma > \tau$ in I, if N_α^σ is a collapsing mouse for K_α^σ, it is also a collapsing mouse for K_α^τ. Since, by Lemma 5.8, collapsing mice are unique it follows that $N_\alpha^\sigma = N_\alpha^\tau$. Pick $\sigma \in I$ such that, for $\mathcal{V}$-almost all α, $\sigma_\alpha > \mathrm{On} \cap N_\alpha^\tau$. Notice that now $\mathrm{On} \cap N_\alpha^\sigma \geqslant \mathrm{On} \cap K_\alpha^\sigma = \sigma_\alpha > \mathrm{On} \cap N_\alpha^\tau$, contradicting $N_\alpha^\sigma = N_\alpha^\tau$.

Assume that κ_α^τ is not the largest cardinal in K_α^τ. Pick some σ as above. Then $\kappa_\alpha^{\tau\,+K_\alpha^\tau}$ exists, and by elementarity so does $\kappa_\alpha^{\tau\,+K_\alpha^\sigma}$. But since $\mathcal{P}(\kappa_\alpha^\tau) \cap K_\alpha^\tau \subsetneq \mathcal{P}(\kappa_\alpha^\tau) \cap K_\alpha^\sigma$, $\kappa_\alpha^{\tau\,+K_\alpha^\tau} < \kappa_\alpha^{\tau\,+K_\alpha^\sigma}$, contradicting $K_\alpha^\tau \prec K_\alpha^\sigma$. □(Claim 2)

Claim 3 κ^+ is a successor cardinal in W.

Proof This is immediate by the elementarity of π_α^τ. □(Claim 3)

So we have $\kappa^+ = \lambda^{+W}$. For $\alpha \in C_\tau$, let $\lambda_\alpha^\tau = {\pi_\alpha^\tau}^{-1}(\lambda)$. Then $\lambda_\alpha^\tau = \kappa_\alpha^\tau$ is the largest cardinal in K_α^τ.

Claim 4 There is a set $C \in \mathcal{V}$ and a sequence $\langle M_\alpha \mid \alpha \in C\rangle$ of mice of size less than κ such that for all $\tau \in I$, and for $\mathcal{V}$-almost all α, if M_α and K_α^τ coiterate to respectively M_α^* and $K_\alpha^{\tau *}$, then $K_\alpha^{\tau *} \subseteq M_\alpha^*$ and the coiteration is simple on the K-side.

The point of this claim is that the sequence $\langle M_\alpha \mid \alpha \in C\rangle$ is independent of τ.

Proof To prove the claim, we distinguish several different cases.

Case 1 κ is not overlapped, i. e., $\forall \mu < \kappa \left(O^W(\mu) < \kappa\right)$.

Case 1.1 κ is a successor in W, say $\kappa = \vartheta^{+W}$.

Then we must have $\bar{o} := \sup\{O^W(\mu) \mid \mu < \kappa\} < \kappa$: Assume contrariwise that $\bar{o} = \kappa$. Then the measurables μ have to be cofinal in κ, since any measurable μ is not overlapped, i. e., $\mu > O^W(\mu')$ for $\mu' < \mu$. But we assumed κ to be a successor cardinal in W — contradiction.

Let $C := (\max\{\bar{o}, \vartheta\}, \kappa) \in \mathcal{V}$. Then for any $\alpha \in C$ there is a minimal γ_α such that, setting $M_\alpha := W{\downarrow}\gamma_\alpha$, we have

$$M_\alpha \vDash \exists f_\alpha \left(f_\alpha : \vartheta \to \alpha \text{ onto}\right).$$

Pick some arbitrary $\tau \in I$, and let $\alpha \in C_\tau \cap C$. Then $E^W_\kappa = \emptyset$ and thus $E^{K^\tau_\alpha}_\alpha = \emptyset$, too. Coiterate M_α and K^τ_α to get respectively M^*_α and $K^{\tau*}_\alpha$. The iteration is obviously beyond α on both sides. Since α is a cardinal in K^τ_α and $E^{K^\tau_\alpha}_\alpha = \emptyset$, the coiteration is, in fact, above α on the K-side. Let $F := E^{M_\alpha}_\nu$ be the first extender used on the M-side, let $\mu = \text{crit}(F)$. If μ were less than ϑ, then we would have $\nu \geqslant \alpha > \vartheta \geqslant \mu^{+W}$, so F would be a full extender on W and $\nu < O^W(\mu) \leqslant \bar{o} < \alpha$, contradicting the fact that the iteration is beyond α. Thus $\mu \geqslant \vartheta$, i. e., the iteration is above ϑ.

Assume that the coiteration is not simple on the K-side. Then it must be simple on the M-side and $M^*_\alpha \subseteq K^{\tau*}_\alpha$. Assume that $K^{\tau*}_\alpha \nsubseteq M^*_\alpha$. Then $M^*_\alpha \in K^{\tau*}_\alpha$, and again, the coiteration must be simple on the M-side. In both cases, one can conclude that

$$\begin{aligned}\mathcal{P}(\vartheta) \cap M_\alpha &= \mathcal{P}(\vartheta) \cap M^*_\alpha \\ &\subseteq \mathcal{P}(\vartheta) \cap K^{\tau*}_\alpha \subseteq \mathcal{P}(\vartheta) \cap K^\tau_\alpha.\end{aligned}$$

But this implies $K^\tau_\alpha \vDash \alpha \notin \text{Card}$, a contradiction. Thus the coiteration must be simple on the K-side and $K^{\tau*}_\alpha \subseteq M^*_\alpha$, as claimed.

Case 1.2 κ is a limit cardinal in W.

In this case set $C := \text{Card}^W \cap \kappa \in \mathcal{V}$. For $\alpha \in C$, there exists $\gamma_\alpha < \kappa$ such that $\mathcal{P}(\alpha) \cap W \in W{\downarrow}\gamma_\alpha =: M_\alpha$.

Let $\tau \in I$ and pick $\alpha \in C_\tau \cap C$. Again we must have $E^W_\kappa = \emptyset$, and thus $E^{K^\tau_\alpha}_\alpha = \emptyset$. As before, the coiteration is thus above (and not only beyond) α on the K-side.

In fact, the coiteration will also be above α on the M-side for $\mathcal{V}$-almost all α. Otherwise, some $E^{M_\alpha}_\xi$ with critical point less than α and index $\xi \geqslant \alpha$ will be used. By coherency, and since $\alpha \in \mathrm{Card}^W$, $E^{M_\alpha}_\alpha = E^W_\alpha$ is also an extender, with the same critical point μ_α, say. Then the function mapping α to μ_α is regressive and monotone increasing on a set in $\mathcal{V}$, hence constant on a set in $\mathcal{V}$. But then one concludes that for some $\mu < \kappa$, $O^W(\mu) \geqslant \kappa$, contradicting the assumption that κ is not overlapped.

So assume w.l.o.g. that the coiteration is above α for all $\alpha \in C_\tau \cap C$, and pick any such α. Let $K^{\tau *}_\alpha$ and M^*_α be the coiterates of respectively K^τ_α and M_α. As in the first case, if either the coiteration is not simple on the K-side or if $K^{\tau *}_\alpha \nsubseteq M^*_\alpha$, then the coiteration must be simple on the M-side and $M^*_\alpha \subseteq K^{\tau *}_\alpha$. Then

$$\begin{aligned}\mathcal{P}(\alpha) \cap W = \mathcal{P}(\alpha) \cap M_\alpha &= \mathcal{P}(\alpha) \cap M^*_\alpha \\ &\subseteq \mathcal{P}(\alpha) \cap K^{\tau *}_\alpha \subseteq \mathcal{P}(\alpha) \cap K^\tau_\alpha.\end{aligned}$$

Note that, since κ is not overlapped in W, $O^W(\alpha) < \kappa = \pi^\tau_\alpha(\alpha)$. Thus one can define an extender at α, $O^W(\alpha)$ on W as in Case 1.1 of Claim 1, and reach exactly the same contradiction as there.

Hence, for $\mathcal{V}$-almost all α, the coiteration is simple on the K-side and $K^{\tau *}_\alpha \subseteq M^*_\alpha$.

Case 2 κ is overlapped in W

Then there is some $\mu < \kappa$ such that $O^W(\mu) \geqslant \kappa$. If κ were a limit cardinal in W then, since κ is also regular (in V and hence in W), we would get $W{\downarrow}\kappa \models \mathrm{ZFC}$ and thus get a set model for a strong cardinal.

So κ is a successor in W, say $\kappa = \vartheta^{+W}$, as in Case 1.1. We can assume that for all $\tau \in I$ and all $\alpha \in C_\tau$, $\alpha > \vartheta$.

Case 2.1 $\mu = \vartheta$.

Let $C := (\vartheta, \kappa) \in \mathcal{V}$. As in Case 1.1, for any $\alpha \in C$ there is a minimal γ_α such that, setting $M_\alpha := W{\downarrow}\gamma_\alpha$, we have

$$M_\alpha \vDash \exists f_\alpha \left(f_\alpha : \vartheta \to \alpha \text{ onto}\right).$$

Let $\tau \in I$, $\alpha \in C_\tau \cap C$, and coiterate K^τ_α and M_α to get $K^{\tau *}_\alpha$ and M^*_α. Then the coiteration is beyond α and thus above ϑ on both sides. Now conclude, literally as in Case 1.1, that the coiteration must be simple on the K-side and $K^{\tau *}_\alpha \subseteq M^*_\alpha$, as claimed.

Case 2.2 $\mu < \vartheta$.

In this case we must have $O^W(\mu) < \kappa^{+W}$. Otherwise, pick the γ_α as in Case 2.1. If $O^W(\mu) \geqslant \kappa^{+W}$, then, for any $\tau \in I$ and any $\alpha \in C_\tau$, $\alpha > \vartheta$, one has $O^{K^\tau_\alpha}(\mu) = \tau_\alpha$, and by the Condensation Lemma $K^\tau_\alpha = W \downarrow \tau_\alpha$. If τ is now chosen so that for $\mathcal{V}$-almost all α, $\tau_\alpha \geqslant \gamma_\alpha$, one gets the contradiction that there is, in K^τ_α, a function f_α mapping ϑ onto α.

Let $\bar\tau := \min(I)$, $C := C_{\bar\tau}$. For $\alpha \in C$, let $\nu_\alpha := O^{K^{\bar\tau}_\alpha}(\mu)$, and set $M_\alpha := \mathrm{Ult}(W|\nu_\alpha, E^W_{\nu_\alpha})$.

Let $\tau \in I$, and let $\alpha \in C_{\bar\tau\tau} \cap C$. Then, since $O^{K^{\bar\tau}_\alpha}(\mu) = \nu_\alpha < \bar\tau_\alpha$ and $K^{\bar\tau}_\alpha \prec K^\tau_\alpha$, $O^{K^\tau_\alpha}(\mu) = \nu_\alpha$, and by the Condensation Lemma $K^\tau_\alpha|\nu_\alpha = W{\downarrow}\nu_\alpha$, so that also $M_\alpha|\nu_\alpha = K^\tau_\alpha|\nu_\alpha$.

The assumption that $\mathcal{P}(\nu_\alpha) \cap M_\alpha \subseteq \mathcal{P}(\nu_\alpha) \cap K^\tau_\alpha$ leads to a contradiction, literally as in the second part of Case 2.2.2.1 of the proof of Claim 1. All of the relevant assumptions of that case hold here, too.

So we can conclude that $\mathcal{P}(\nu_\alpha) \cap M_\alpha \nsubseteq \mathcal{P}(\nu_\alpha) \cap K^\tau_\alpha$. Coiterate K^τ_α and M_α, yielding respectively M^*_α and $K^{\tau *}_\alpha$. Notice that the coiteration is above ν_α on both sides: The coiteration is above μ and beyond ν_α, and $\nu_\alpha = O^{M_\alpha}(\mu) = O^{K^\tau_\alpha}(\mu)$ — we are assuming that there are no overlapping extender sequences, so ν_α cannot be a critical point on either side.

Assume that the K-side of the coiteration is non-simple or that $K^{\tau *}_\alpha \nsubseteq M^*_\alpha$. Then the M-side must be simple and $M^*_\alpha \subseteq K^{\tau *}_\alpha$. But then

$$\begin{aligned}\mathcal{P}(\nu_\alpha) \cap M_\alpha &= \mathcal{P}(\nu_\alpha) \cap M^*_\alpha \\ &\subseteq \mathcal{P}(\nu_\alpha) \cap K^{\tau *}_\alpha \subseteq \mathcal{P}(\nu_\alpha) \cap K^\tau_\alpha,\end{aligned}$$

which we concluded not to hold. So if $\mathcal{P}(\nu_\alpha) \cap M_\alpha \not\subseteq \mathcal{P}(\nu_\alpha) \cap K^\tau_\alpha$, then the coiteration must be simple on the K-side and $K^{\tau *}_\alpha \subseteq M^*_\alpha$, as desired. $\square$(Claim 4)

Take the set $C \in \mathcal{V}$ and the sequence $\langle M_\alpha \mid \alpha \in C \rangle$ guaranteed by Claim 4. For $\tau \in I$, assume w.l.o.g. that the conclusion of that claim holds for all $\alpha \in C_\tau$. Denote the coiteration of K^τ_α and M_α by respectively $\mathcal{I}^\tau_\alpha$ and $\mathcal{J}^\tau_\alpha$, and let β^τ_α be the length of this coiteration and let $i^{\tau(\xi,\zeta)}_\alpha$, $j^{\tau(\xi,\zeta)}_\alpha$ denote the iteration maps, for $\xi \leqslant \zeta \leqslant \beta^\tau_\alpha$. Let $\mu^{\tau(\xi)}_\alpha$ be the critical point at stage $\xi < \beta^\tau_\alpha$ (cf. Figure 7.5). Recall that $\lambda^\tau_\alpha = {\pi^\tau_\alpha}^{-1}(\lambda)$ is the largest cardinal of K^τ_α, and $\kappa = \lambda^{+W}$.

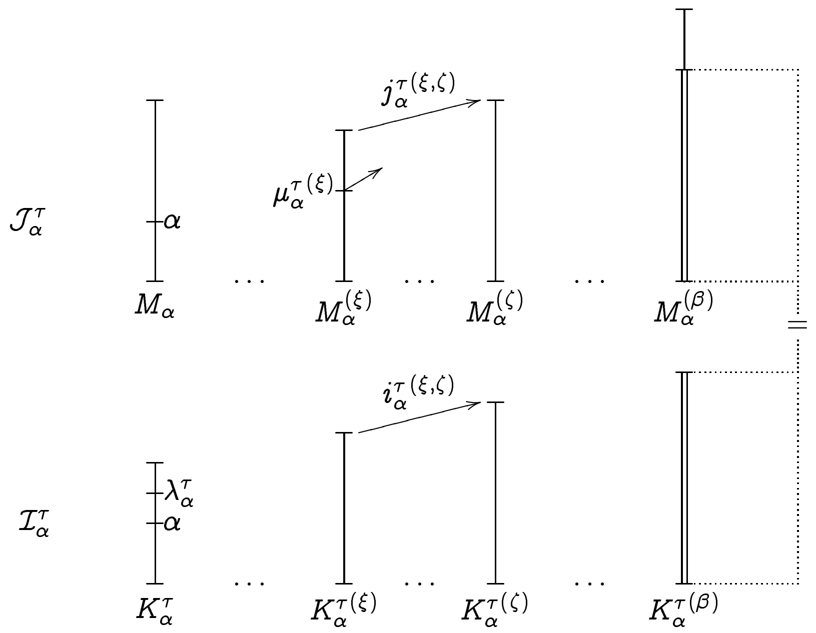

Figure 7.5: The coiteration of M_α and K^τ_α, as guaranteed by Claim 4. $\mathcal{I}^\tau_\alpha$ is a simple iteration, no truncations are performed.

Claim 5 There is a $\tau \in I$ such that for $\mathcal{V}$-almost all α, there is $\xi < \beta^\tau_\alpha$ such that $\mu^{\tau(\xi)}_\alpha \geqslant i^{\tau(0,\xi)}_\alpha(\lambda^\tau_\alpha)$.

Proof First, pick some $\tau \in I$ and let $\alpha \in C_\tau$. To enhance legibility, we drop the indices τ and α occasionally. E. g., let $\mathcal{I}$, $\mathcal{J}$ be the coiteration of K^τ_α and M_α, and let $\mu^{(\xi)}$ be the critical point at stage ξ, where $\xi < \beta = \beta^\tau_\alpha$, the length of the coiteration. Since $K^{\tau(\beta)}_\alpha \subseteq M^{(\beta)}_\alpha$, it follows that $E^{M^{(\beta)}_\alpha} \restriction \delta_\alpha = E^{K^{\tau(\beta)}_\alpha}$, where $\delta_\alpha := \mathrm{On} \cap K^{\tau(\beta)}_\alpha$. Let $\lambda^{\tau(\beta)}_\alpha := i^{(0,\beta)}(\lambda^\tau_\alpha)$. Then $\lambda^{\tau(\beta)}_\alpha$ is the largest cardinal in $K^{\tau(\beta)}_\alpha$.

We can assume that for $\xi < \beta$, $\mu^{(\xi)} < \lambda^{\tau(\xi)}_\alpha$. As in all coiterations, $\beta < \max(\operatorname{card} K^\tau_\alpha, \operatorname{card} M_\alpha)^+ = \kappa$. Thus there exists $\sigma \in I$ such that for $\mathcal{V}$-almost all $\alpha \in C_{\tau\sigma}$, $\sigma_\alpha > \mathrm{On} \cap M^{(\beta)}_\alpha$ (cf. Figure 7.6).

For such α, consider next the coiteration of K^σ_α and M_α, denoted by $\mathcal{I}'$, $\mathcal{J}'$ etc. Then $\mathcal{I}$ is an initial segment of $\mathcal{I}'$. This is proved by a straightforward induction on $\xi \leqslant \beta =$ length of the iteration $\mathcal{I}$, using the fact that the coiterations are simple on K-side in both cases, so that no truncations have to be taken into account there. For the successor step, use [Koe89, Lemma 14.2], which applies here as $K^\tau_\alpha \prec K^\sigma_\alpha$.

Then $\mathrm{On} \cap K^{\sigma(\beta)}_\alpha \geqslant \sigma_\alpha > \mathrm{On} \cap M^{(\beta)}_\alpha$, so $\beta' > \beta$. But actually nothing happens on the K-side: Assume for a contradiction that some extender F is used to build an ultrapower of $K^{\sigma(\beta)}_\alpha$. This extender would have to have index at least $\delta_\alpha = \mathrm{On} \cap K^{\tau(\beta)}_\alpha$. As $\lambda^{\tau(\beta)}_\alpha$ is the largest cardinal in $K^{\tau(\beta)}_\alpha$ and in $K^{\sigma(\beta)}_\alpha$ and no truncations are allowed on the K-side, the critical point of F would have to be less than $\lambda^{\tau(\beta)}_\alpha$. So by coherency $K^{\tau(\beta)}_\alpha$ has extenders with this critical point all the way up to δ_α. Then by elementarity, K^τ_α has extenders with some critical point $\mu < \lambda^\tau_\alpha$, all the way up to τ_α. Thus by the definition of $\kappa^\tau_\alpha = \kappa^{K^\tau_\alpha}$, one gets $\kappa^\tau_\alpha = \mu < \lambda^\tau_\alpha$, contradicting Claim 2.

Effectively, the coiteration on the K-side halts, and all that happens is that $M^{(\beta)}_\alpha$ is iterated till we have agreement. The index $\nu^{(\beta)}$ of the extender used in the next step has to be at least δ_α, of course.

If the coiteration of K^σ_α and M_α terminates after the $(\beta+1)$-th step, find $\varrho \in I$ to make sure that the coiteration of K^ϱ_α and M_α goes at least one step further, by the same argument that gave σ in the first place. Assume w.l.o.g. that already the coiteration of K^σ_α and M_α takes at least $(\beta+2)$-many steps. But again, nothing happens on the K-side after the β-th step, and so $i^{(0,\beta+1)}(\lambda^\sigma_\alpha) = i^{(0,\beta)}(\lambda^\tau_\alpha)$. Fine coiterations are normal, so $\mu^{(\beta+1)} > \nu^{(\beta)} \geqslant \delta_\alpha > i^{(0,\beta)}(\lambda^\tau_\alpha) = i^{(0,\beta+1)}(\lambda^\sigma_\alpha)$, so σ and $\beta+1$ are as desired. $\square$(Claim 5)

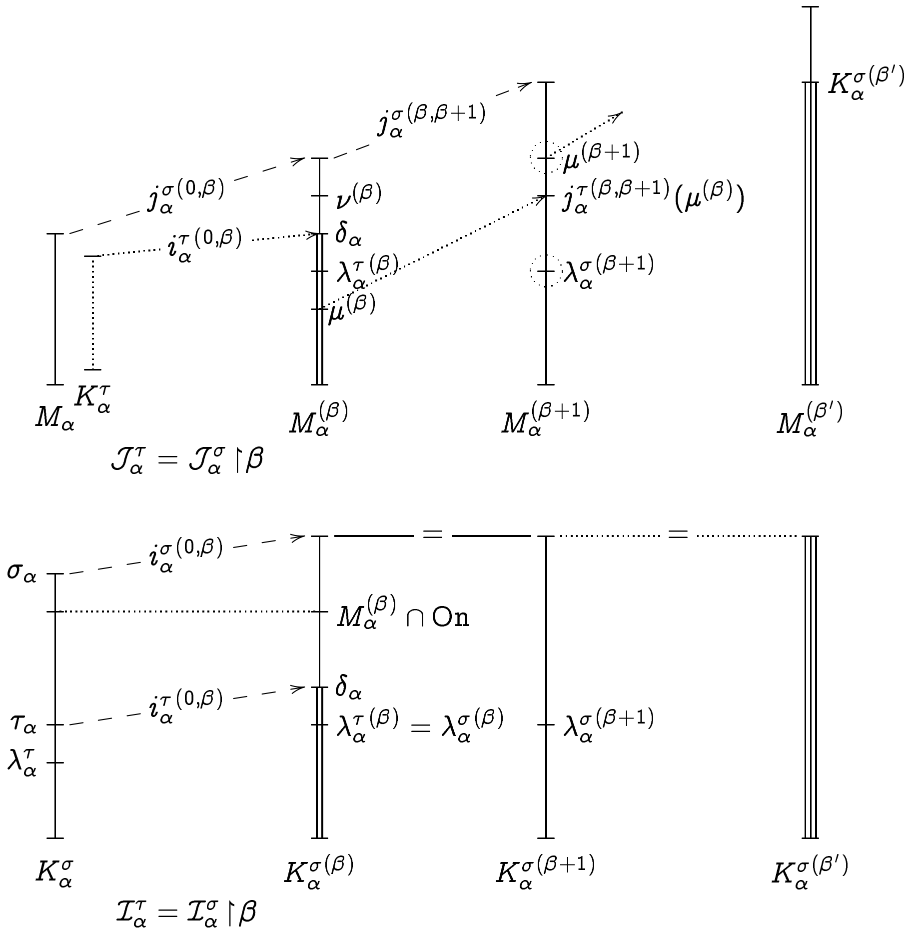

Figure 7.6: By choosing an appropriate σ one can ensure that at some point of the coiteration of K_α^σ and M_α the critical point $\mu^{(\xi)}$ is greater than the largest cardinal $\lambda_\alpha^{\sigma(\xi)}$ of $K_\alpha^{\sigma(\xi)}$. It happens here at $\xi = \beta + 1$ (see the dotted circles), where β is the length of the coiteration of K_α^τ and M_α, and where σ was chosen such that σ_α is greater than $M_\alpha^{(\beta)} \cap \mathrm{On}$. β' denotes the length of the coiteration of K_α^σ and M_α.

This now quickly leads to a contradiction: $\mu^{(\xi)}$ is a cardinal in $M_\alpha^{(\beta)}$. By choosing a suitable σ, one can arrange that $\mathrm{On} \cap K_\alpha^{\sigma(\xi)} > \mu^{(\xi)}$, and hence $K_\alpha^{\sigma(\xi)}$ (and so $K_\alpha^{\sigma(\beta)}$) thinks $\mu^{(\xi)}$ has cardinality $\lambda_\alpha^{\sigma(\xi)} < \mu^{(\xi)}$. This contradicts $K_\alpha^{\sigma(\beta)} \subseteq M_\alpha^{(\beta)}$ (for $\mathcal{V}$-almost all α).

This concludes the discussion of Case A. $\square$(Case A)

Case B Recall that we had proved in Claim 1 that w.l.o.g. for all $\tau \in I$ and all $\alpha \in C_\tau$, K^τ_α is not weakly full. We now want to lead the assumption that $\kappa^{K^\tau_\alpha} = \tau_\alpha$ to a contradiction. As before, by elementarity, if this is true for one $\tau \in I$ and $\mathcal{V}$-almost all $\alpha \in C_\tau$, then it is true for all $\sigma \in I$ (and $\mathcal{V}$-almost all $\alpha \in C_\sigma$). As before, K^τ_α is neat and thus prestrong. As K^τ_α is not weakly full, it cannot be strong, since strong mice *are* weakly full by [Sch96, Lemma 5.9]. Thus there exists a mouse M^τ_α end-extending K^τ_α such that $K^\tau_\alpha \not\subseteq \mathrm{core}(M^\tau_\alpha)$. We aim to show that, given some α, many of these core mice for varying τ are identical. This allows us to conclude that the iteration from this core to some suitable M^τ_α must be long (in the sense of Definition 6.1). This in turn is necessary for the Gitik Game to go through, which will be be final step towards the desired contradiction.

We have to proceed with some care in the choice of M^τ_α.

Claim 6 For each $\tau \in I$ and each $\alpha \in C_\tau$, there is a mouse M^τ_α end-extending K^τ_α such that $K^\tau_\alpha \not\subseteq \bar{M}^\tau_\alpha := \mathrm{core}(M^\tau_\alpha)$, $\mathcal{P}\left(\varrho^\omega_{M^\tau_\alpha}\right) \cap M^\tau_\alpha \subseteq K^\tau_\alpha$ and such that for no intermediate stage M' in the iteration from $\bar{M}^\tau_\alpha$ to M^τ_α does $K^\tau_\alpha \subseteq M'$. Let ϱ^τ_α denote $\varrho^\omega_{M^\tau_\alpha}$.

Proof Let M be any mouse end-extending K^τ_α such that $K^\tau_\alpha \not\subseteq \mathrm{core}(M) =: \bar{M}$. Such M exists as K^τ_α is not weakly full. We want to arrange

$$\mathcal{P}\left(\varrho^\omega_M\right) \cap M \subseteq K^\tau_\alpha.$$

Note that $\varrho^\omega_M < \tau_\alpha$. Otherwise, since the iteration from $\bar{M}$ to M is simple and above ϱ^ω_M, we would have $K^\tau_\alpha \subseteq M{\downarrow}\tau_\alpha = \bar{M}{\downarrow}\tau_\alpha \subseteq \bar{M}$, a contradiction. Let

$$\eta := \max\{\xi \leqslant \mathrm{On} \cap M \mid \mathcal{P}\left(\varrho^\omega_M\right) \cap M{\downarrow}\xi \subseteq K^\tau_\alpha\}.$$

Let $N := M{\downarrow}\eta$. Then surely N is a mouse and $K^\tau_\alpha \subseteq N$. Note that $\varrho^\omega_N \leqslant \varrho^\omega_M$ by the choice of η. Thus $\mathcal{P}\left(\varrho^\omega_N\right) \cap N \subseteq K^\tau_\alpha$.

If $K^\tau_\alpha \not\subseteq \mathrm{core}(N)$, then we have a good candidate for M^τ_α: Let M^τ_α be the first stage of the iteration from $\mathrm{core}(N)$ to N such that $K^\tau_\alpha \subseteq M^\tau_\alpha$. i. e., if $N^{(0)} = \mathrm{core}(N)$, $N^{(\vartheta)} = N \supseteq K^\tau_\alpha$, then let $\xi \leqslant \vartheta$ be minimal such that $N^{(\xi)} \supseteq K^\tau_\alpha$ and set $M^\tau_\alpha := N^{(\xi)}$. Since the iteration is simple and above ϱ^ω_N, one has $\varrho^\omega_{M^\tau_\alpha} = \varrho^\omega_N$ and $\mathcal{P}\left(\varrho^\omega_{M^\tau_\alpha}\right) \cap M^\tau_\alpha = \mathcal{P}\left(\varrho^\omega_N\right) \cap N \subseteq K^\tau_\alpha$, and $K^\tau_\alpha \not\subseteq \mathrm{core}(M^\tau_\alpha) =: \bar{M}^\tau_\alpha$.

Otherwise, $K^\tau_\alpha \subseteq \mathrm{core}(N)$. Note that this implies $N \neq M$ and so $\eta < \mathrm{On} \cap M$. We first claim that $\varrho^\omega_N = \varrho^\omega_M$: If $\varrho^\omega_N < \varrho^\omega_M$, then pick some $a \in \mathcal{P}(\varrho^\omega_N) \cap \Sigma^*(N) \setminus N$. As $\eta < \mathrm{On} \cap M$, we conclude

$$\begin{aligned} a \in \mathcal{P}(\varrho^\omega_N) \cap M = \mathcal{P}(\varrho^\omega_N) \cap \bar{M} \\ \subseteq \bar{M}{\downarrow}(\varrho^\omega_N)^{+\bar{M}} \subseteq \bar{M}{\downarrow}\varrho^\omega_M = M{\downarrow}\varrho^\omega_M = K^\tau_\alpha{\downarrow}\varrho^\omega_M \subseteq N, \end{aligned}$$

a contradiction. (This uses the fact that $(\varrho^\omega_N)^{+\bar{M}} \leqslant \varrho^\omega_{\bar{M}} = \varrho^\omega_M$ and that the iteration from $\bar{M}$ to M is simple and above ϱ^ω_M.)

Next we claim that $\eta = \mathrm{On} \cap M$, which immediately gives a contradiction. Otherwise, i. e., if $\eta < \mathrm{On} \cap M$, as before pick some $a \in \mathcal{P}(\varrho) \cap \Sigma^*(N) \setminus N$, where ϱ denotes $\varrho^\omega_N = \varrho^\omega_M$. As in the previous paragraph, $a \in \bar{M}$. Thus either $a \in \bar{M}{\downarrow}\varrho^{+\bar{M}}$ or ϱ is the largest cardinal of $\bar{M}$. In this latter case, since M is a simple iterate of $\bar{M}$ above ϱ, we must have $\bar{M} = M$, contradicting $K^\tau_\alpha \subseteq M$ and $K^\tau_\alpha \not\subseteq \bar{M}$. But the former case also leads to a contradiction: Again as M is a simple iterate of $\bar{M}$ above ϱ, M and $\bar{M}$ agree up to $\varrho^{+M} = \varrho^{+\bar{M}}$. Thus $a \in M \downarrow \varrho^{+M}$. Now if $\varrho^{+M} \leqslant \tau_\alpha$, then $a \in K^\tau_\alpha \subseteq N$, contradiction. But if $\varrho^{+M} > \tau_\alpha$, then $K^\tau_\alpha \subseteq M{\downarrow}\varrho^{+M} = \bar{M}{\downarrow}\varrho^{+M}$, also a contradiction. □(Claim 6)

Claim 7 Let $\tau < \sigma$ both be elements of I and let $\alpha \in C_{\tau\sigma}$. Then either

$$\bar{M}^\sigma_\alpha \leqslant^* \bar{M}^\tau_\alpha \qquad \text{or} \qquad \varrho^\sigma_\alpha < \varrho^\tau_\alpha,$$

where $\leqslant^*$ denotes the canonical pre-well-ordering of mice.

Proof Assume that $\varrho^\sigma_\alpha \geqslant \varrho^\tau_\alpha$. We first claim that the coiteration of $\bar{M}^\tau_\alpha$ and $\bar{M}^\sigma_\alpha$ is above $\varrho := \min\{\varrho^\sigma_\alpha, \varrho^\tau_\alpha\} = \varrho^\tau_\alpha$. Otherwise, since coiterations are normal, there is a first disagreement between $\bar{M}^\tau_\alpha$ and $\bar{M}^\sigma_\alpha$ at some μ, ξ, where $\mu < \varrho$.

Say $E^{\bar{M}^\tau_\alpha}_\xi$ is at μ, ξ. Since $\mathcal{I}_{\bar{M}^\tau_\alpha, M^\tau_\alpha}$ is above $\varrho^\tau_\alpha = \varrho$, it follows that $E^{M^\tau_\alpha}_\xi = E^{\bar{M}^\tau_\alpha}_\xi$. If $\xi < \tau_\alpha$, then

$$E^{\bar{M}^\tau_\alpha}_\xi = E^{K^\tau_\alpha}_\xi = E^{K^\sigma_\alpha}_\xi = E^{M^\sigma_\alpha}_\xi = E^{\bar{M}^\sigma_\alpha}_\xi,$$

where the last equality holds as $\mathcal{I}_{\bar{M}^\sigma_\alpha, M^\sigma_\alpha}$ is above $\varrho^\sigma_\alpha \geqslant \varrho$. But this is a contradiction. So $\xi \geqslant \tau_\alpha$. But then $E^{\bar{M}^\tau_\alpha} \restriction \tau_\alpha = E^{M^\tau_\alpha} \restriction \tau_\alpha = E^{K^\tau_\alpha}$, contradicting the choice of M^τ_α.

Thus $E_\xi^{\bar{M}_\alpha^\tau} = \emptyset$, and $E_\xi^{\bar{M}_\alpha^\sigma}$ is at μ, ξ. As before, $E_\xi^{\bar{M}_\alpha^\sigma} = E_\xi^{M_\alpha^\sigma}$, $E_\xi^{\bar{M}_\alpha^\tau} = E_\xi^{M_\alpha^\tau}$ and ξ must be greater than τ_α. By the argument of the previous paragraph, ξ must be less than σ_α. Thus $E_\xi^{K_\alpha^\sigma}$ is at μ, ξ, and hence by coherence $E_{\tau_\alpha}^{K_\alpha^\sigma}$ is at μ, τ_α, and $E^{K_\alpha^\tau} \restriction \tau_\alpha = E^{K_\alpha^\sigma} \restriction \tau_\alpha$ (recall that K_α^τ does not have a largest cardinal, so $\mu^{+K_\alpha^\tau} = \mu^{+K_\alpha^\sigma}$ exists and is less than τ_α). But this implies $\kappa_\alpha^\tau = \kappa^{K_\alpha^\tau} = \mu < \tau_\alpha$, whereas the general assumption of Case B is that $\kappa_\alpha^\tau = \tau_\alpha$.

Thus the coiteration of $\bar{M}_\alpha^\tau$ and $\bar{M}_\alpha^\sigma$ is above $\varrho_\alpha^\tau \leqslant \varrho_\alpha^\sigma$. Let $\widetilde{M}_\alpha^\tau$ and $\widetilde{M}_\alpha^\sigma$ be the comparable coiterates and assume $\widetilde{M}_\alpha^\tau \in \widetilde{M}_\alpha^\sigma$. Then

$$\begin{aligned}
\mathcal{P}(\varrho_\alpha^\tau) \cap \Sigma^*(M_\alpha^\tau) &= \mathcal{P}(\varrho_\alpha^\tau) \cap \Sigma^*(\bar{M}_\alpha^\tau) && \text{as } \mathcal{I}_{M_\alpha^\tau, \bar{M}_\alpha^\tau} \text{ is simple above } \varrho_\alpha^\tau \\
&= \mathcal{P}(\varrho_\alpha^\tau) \cap \Sigma^*(\widetilde{M}_\alpha^\tau) && \text{as } \mathcal{I}_{\bar{M}_\alpha^\tau, \widetilde{M}_\alpha^\tau} \text{ is simple above } \varrho_\alpha^\tau \\
&\subseteq \mathcal{P}(\varrho_\alpha^\tau) \cap \widetilde{M}_\alpha^\sigma && \text{as } \widetilde{M}_\alpha^\tau \in \widetilde{M}_\alpha^\sigma \\
&\subseteq \mathcal{P}(\varrho_\alpha^\tau) \cap \bar{M}_\alpha^\sigma && \text{as } \mathcal{I}_{\bar{M}_\alpha^\sigma, \widetilde{M}_\alpha^\sigma} \text{ is above } \varrho_\alpha^\sigma \geqslant \varrho_\alpha^\tau \\
&= \mathcal{P}(\varrho_\alpha^\tau) \cap M_\alpha^\sigma && \text{as } \mathcal{I}_{\bar{M}_\alpha^\sigma, M_\alpha^\sigma} \text{ is simple above } \varrho_\alpha^\sigma \\
&= \mathcal{P}(\varrho_\alpha^\tau) \cap K_\alpha^\sigma && \text{as } \mathcal{P}(\varrho_\alpha^\sigma) \cap M_\alpha^\sigma \subseteq K_\alpha^\sigma \\
&= \mathcal{P}(\varrho_\alpha^\tau) \cap K_\alpha^\tau \\
&\subseteq M_\alpha^\tau,
\end{aligned}$$

a contradiction! The last equation holds since K_α^τ has no largest cardinal: Let $\xi < \tau_\alpha$. Then $\xi^{+K_\alpha^\tau}$ exists and is less than τ_α, so $\mathcal{P}(\xi) \cap K_\alpha^\tau \in K_\alpha^\tau$ and as $K_\alpha^\tau \prec K_\alpha^\sigma$, $\mathcal{P}(\xi) \cap K_\alpha^\tau = \mathcal{P}(\xi) \cap K_\alpha^\sigma$.

Thus the assumption $\widetilde{M}_\alpha^\tau \in \widetilde{M}_\alpha^\sigma$ was wrong and $\widetilde{M}_\alpha^\sigma \subseteq \widetilde{M}_\alpha^\tau$ after all, i. e., $\bar{M}_\alpha^\sigma \leqslant^* \bar{M}_\alpha^\tau$. □(Claim 7)

We are now in a position to prove that all the prerequisites of the Gitik Game are satisfied by some K_α^σ.

CLAIM 8 There exists a $\sigma \in I$ and an $\alpha \in C_\sigma$ (in fact, unboundedly many such α) such that the iteration $\mathcal{I}_{\bar{M}_\alpha^\sigma, M_\alpha^\sigma}$ from $\bar{M}_\alpha^\sigma$ to M_α^σ is not short (in the sense of Definition 6.1).

PROOF We will inductively choose a κ-long sequence of elements of I, corresponding sets in $\mathcal{V}$ and functions from ${}^\kappa\kappa$.

Let τ^0 be arbitrary, and set $A^0 := C_{\tau^0} \in \mathcal{V}$, $f^0 \in {}^\kappa\kappa$, where

$$f^0(\alpha) = \begin{cases} \max\{s^{(\omega_1)}(\bar{M}_\alpha^{\tau^0}), \tau^0_\alpha\} + 1 & \text{if } \alpha \in A^0, \\ 0 & \text{else.} \end{cases}$$

By Lemma 6.2, f^0 really takes values less than κ.

Assume $\langle \tau^i \mid i < \zeta \rangle$ had been chosen. Define $h^\zeta \in {}^\kappa\kappa$ by setting

$$h^\zeta(\alpha) = \sup\{f^i(\alpha) + 1 \mid i < \alpha \cap \zeta\}.$$

Then since inductively all the f^i take values less than κ, so does h^ζ. Using Lemma 7.9 find $\tau^\zeta \in I$ such that $\tau^\zeta > \sup\{\tau^i \mid i < \zeta\}$ and such that for $\mathcal{V}$-almost all α $\tau^\zeta_\alpha > h^\zeta(\alpha)$. Set $A^\zeta := \{\alpha \in C_{\tau^\zeta} \mid \tau^\zeta > h^\zeta(\alpha)\} \in \mathcal{V}$ and let

$$f^\zeta(\alpha) = \begin{cases} \max\{s^{(\omega_1)}(\bar{M}_\alpha^{\tau^\zeta}), \tau^\zeta_\alpha\} + 1 & \text{if } \alpha \in A^\zeta, \\ 0 & \text{else.} \end{cases}$$

Since $\mathcal{V}$ is not (ϱ, κ)-regular and $\varrho \geqslant \omega_1$, it is not weakly (ω, κ)-regular by Lemma 7.3. Thus $\langle A^i \mid i < \kappa \rangle$ is *not* a weak regularity sequence, and so there are indices $\langle i_j \mid j \in \omega \rangle$ such that $\bigcap_{j \in \omega} A^{i_j}$ is unbounded in κ.

Recall that $C_{\tau\sigma}$ is a final segment of C_σ. So let $\beta_{\tau\sigma}$ be (least) such that

$$\forall \alpha > \beta_{\tau\sigma} \left(\alpha \in C_\tau \cap C_\sigma \;\rightarrow\; \alpha \in C_{\tau\sigma}\right).$$

Let, for $j < k < \omega$, $\beta_{jk} := \beta_{\tau^{i_j}\tau^{i_k}}$ and set

$$\bar{\alpha} := \sup\{i_j \mid j < \omega\} \cup \sup\{\beta_{jk} \mid j < k < \omega\} < \kappa.$$

Now pick some $\alpha \in \left(\bigcap_{j \in \omega} A^{i_j}\right) \setminus \bar{\alpha}$.

Let $j < k < \omega$. Then $\alpha > i_k$, so $h^{i_j}(\alpha) > f^{i_k}(\alpha) > \max\{s^{(\omega_1)}(\bar{M}_\alpha^{\tau^{i_k}}), \tau^{i_k}_\alpha\}$, as also $\alpha \in A^{i_k}$. Thus

$$\tau^{i_j}_\alpha > h^{i_j}(\alpha) > \tau^{i_k}_\alpha \cup s^{(\omega_1)}(\bar{M}_\alpha^{\tau^{i_k}}).$$

and so by Claim 7

$$\forall j < k < \omega \left(\underbrace{\bar{M}_\alpha^{\tau^{i_j}} = \bar{M}_\alpha^{\tau^{i_k}}}_{\text{(A)}} \text{ or } \underbrace{\bar{M}_\alpha^{\tau^{i_j}} >^* \bar{M}_\alpha^{\tau^{i_k}}}_{\text{(B)}} \text{ or } \underbrace{\varrho_\alpha^{\tau^{i_j}} > \varrho_\alpha^{\tau^{i_k}}}_{\text{(C)}}\right).$$

By Ramsey's Theorem, there must be a homogeneous set $J \subseteq \omega$ of size ω such that one of (A), (B), (C) holds for all $j < k \in J$. Obviously, if J were homogeneous of "type (C)", then we would get an infinite descending sequence of ordinals. Similarly, a homogeneous set of "type (B)" would give an infinite descending sequence of mice, also a contradiction. Thus there must be ω-many indices τ^i that share the same core mouse $\bar{M}$. In fact, we are only interested in two of them. I. e., we have $\tau, \sigma \in I$ such that

$$\mathrm{On} \cap M_\alpha^\sigma \geqslant \sigma_\alpha > h^\sigma(\alpha) > \max\{s^{(\omega_1)}(\bar{M}_\alpha^\tau), \tau_\alpha\}.$$

and $\bar{M}_\alpha^\sigma = \bar{M}_\alpha^\tau$. But then the iteration $\mathcal{I}_{\bar{M}_\alpha^\sigma, M_\alpha^\sigma}$ from $\bar{M}_\alpha^\sigma$ to M_α^σ cannot be short. □(Claim 8)

We are now ready to play the Gitik Game. Our presentation follows closely that of [Sch96, Chapter 6].

Pick σ as in Claim 8 and some α such that the iteration $\mathcal{I}_{\bar{M},M}$ from $\bar{M} := \bar{M}_\alpha^\sigma$ to $M := M_\alpha^\sigma$ is not short. Let $\mathcal{I}_{\bar{M},M}$ have length ϑ, iteration maps $\langle \tilde{\pi}_{ij} \mid i \leqslant j \leqslant \vartheta \rangle$, and indices $\langle \tilde{\nu}_i \mid i < \vartheta \rangle$. As the iteration is not short, there is a sequence $\langle i_\xi \mid \xi \leqslant \omega_1 \rangle$ of ordinals less than ϑ such that

$$\forall \xi \leqslant \zeta \leqslant \omega_1 \left(\tilde{\pi}_{i_\xi i_\zeta}(\tilde{\nu}_{i_\xi}) = \tilde{\nu}_{i_\zeta} \right).$$

Let us introduce some further notational simplifications. We will subsequently only be interested in stages of the iteration determined by the sequence $\langle i_\xi \mid \xi \leqslant \omega_1 \rangle$, thus set, for $\xi \leqslant \zeta \leqslant \omega_1$,

$$N^\xi := M^{i_\xi} \qquad \nu_\xi := \tilde{\nu}_{i_\xi}$$
$$\pi_{\xi\zeta} := \tilde{\pi}_{i_\xi i_\zeta} \qquad \mu_\xi := \mathrm{crit}(E_{\nu_\xi}^{N^\xi})$$

Let $F := E_{\nu_{\omega_1}}^{N^{\omega_1}}$. The point of the Gitik Game is to reconstruct this extender inside the collapsed substructure $\bar{H} := H_\alpha^\sigma$. (For convenience, denote K_α^σ by $\bar{K}$, too.)

Claim 9 $F \in \bar{H}$.

Before we proceed with the proof of this claim, let us show how to conclude the discussion of Case (B) and hence the proof of the theorem, given Claim 9.

Claim 10 F is countably complete.

Proof Let $\mu := \mu_{\omega_1}$, $\nu := \nu_{\omega_1}$. Let $\langle a_n \mid n\in\omega\rangle$ be a sequence of elements from $[\nu]^{<\omega}$ and let $\langle x_n \mid n \in \omega\rangle$ be a sequence of sets such that for all $n \in \omega$, $x_n \in F_{a_n}$. We have to show that there is an orderpreserving map $t : \bigcup_{n\in\omega} a_n \to \mu$ such that for all $n\in\omega$, $t'' a_n \in x_n$.

First, find $i < \omega_1$ such that there a sets y_n and b_n, $n\in\omega$, in N^i such that

$$x_n = \pi_{i\omega_1}(y_n) \qquad\qquad a_n = \pi_{i\omega_1}(b_n).$$

Let $t := \pi_{i\omega_1}^{-1} \restriction \bigcup_{n\in\omega} a_n$. Then t is an orderpreserving map from $\bigcup_{n\in\omega} a_n$ to $\bigcup_{n\in\omega} b_n$, since $\pi_{i\omega_1}$ is elementary. Also note that the iteration $\mathcal{I}_{N^0 N^{\omega_1}}$ is normal, so $\mu > \nu^i$. $\bigcup_{n\in\omega} a_n \subseteq \nu$ and $\pi_{i\omega_1}(\nu_i) = \nu$, so $\bigcup_{n\in\omega} b_n \subseteq \nu_i \subseteq \mu$.

Finally, $x_n \in F_{a_n} = \left(E^{N^{\omega_1}}_{\pi_{i\omega_1}(\nu_i)}\right)_{\pi_{i\omega_1}(b_n)}$, thus $y_n \in \left(E^{N^i}_{\nu_i}\right)_{b_n}$. By the normality of $\mathcal{I}_{N^0 N^{\omega_1}}$, $b_n \in \pi_{i\omega_1}(y_n)$, i. e., $t'' a_n = b_n \in x_n$. □(Claim 10)

Now recall that N^{ω_1} was just a stage in the iteration from $\bar{M}^\sigma_\alpha$ to M^σ_α. We have

$$\begin{aligned} N^{\omega_1}\!\downarrow\!\nu_{\omega_1} &= M^\sigma_\alpha\!\downarrow\!\nu_{\omega_1} && \text{since } \mathcal{I}_{\bar{M}^\sigma_\alpha, M^\sigma_\alpha} \text{ is normal} \\ &= K^\sigma_\alpha\!\downarrow\!\nu_{\omega_1}. \end{aligned}$$

The second equation holds since ν_{ω_1} must be less than σ_α. Otherwise an earlier stage of the iteration $\mathcal{I}_{\bar{M}^\sigma_\alpha, M^\sigma_\alpha}$ would already have satisfied the conditions of Claim 6. Thus we have that

$$H^\sigma_\alpha \models \langle K^\sigma_\alpha\!\downarrow\!\nu_{\omega_1}, F\rangle \text{ is a premouse such that } F \text{ is countably complete.}$$

By definition, $K^\sigma_\alpha = (\mathrm{K}^c)^{H^\sigma_\alpha}$. Thus it follows that $E^{K^\sigma_\alpha}_{\nu_{\omega_1}} = F$. On the other hand, F is an extender used in the iteration from $\bar{M}^\sigma_\alpha$ to M^σ_α, an end-extension of K^σ_α, and thus $E^{K^\sigma_\alpha}_{\nu_{\omega_1}} = E^{M^\sigma_\alpha}_{\nu_{\omega_1}} = \emptyset$, a contradiction.

This concludes the proof of the theorem, save for Claim 9.

Proof of Claim 9

Our aim is to show that the extender $F = E^{N^{\omega_1}}_{\nu_{\omega_1}}$ is an element of the structure $\bar{H} = H^\sigma_\alpha$. By our choice of extenders (or rather indices) we have

$$\forall \xi \leqslant \zeta \leqslant \omega_1 \left(\pi_{\xi\zeta}(\nu_\xi) = \nu_\zeta\right)$$

and so it will be enough to reconstruct in $\bar{H}$ a sufficiently good approximation to the direct limit system $\langle N^\xi \mid \xi \leqslant \omega_1\rangle$. More precisely, it will be enough to define in $\bar{H}$ a *thread* b_β for each $\beta < \nu_{\omega_1}$, where the thread b_β is taken to represent β in the direct limit N^{ω_1}. Thus b_β will be a sequence of ordinals $\langle \beta_i \mid i < \omega_1\rangle$, such that $\beta_i < \mu$ (since $\beta_i < \nu_i < \mu_{i+1} < \mu$) and such that

$$\exists j < \omega_1 \ \forall i \in [j, \omega_1) \ \Big(\pi_{i\omega_1}(\beta_i) = \beta\Big).$$

Given such threads in $\bar{H}$, one can reconstruct F as follows: Let $a \in [\nu_{\omega_1}]^{<\omega}$ and $x \in \mathcal{P}\left([\mu]^{\overline{\overline{a}}}\right) \cap N^{\omega_1}$. Assume $a = \{\beta^1, \ldots, \beta^n\}$. Then x is in F_a if a final sequence of the thread for a is in x, i. e.,

$$x \in F_a \ \leftrightarrow \ \exists j < \omega_1 \ \forall i \in [j, \omega_1) \ \Big(\{\beta_i^1, \ldots, \beta_i^n\} \in x\Big).$$

Thus if the threads b_β are all definable in $\bar{H}$, then F is definable in $\bar{H}$ and hence an element of $\bar{H}$.

The threads will be defined via winning strategies for the second player in the following game, due to Gitik ([Git93]).

The *Gitik Game* $\mathfrak{G}$ is played by two players, say Alice and Bob, who alternatingly make ω-many moves. Alice is the first to play. On her n-th move Alice plays a pair $\langle B_n, h_n\rangle$ such that

$$B_n \subseteq \nu_{\omega_1}, \ \mathrm{card}(B_n) \leqslant \aleph_1, \ \text{ and } B_{n-1} \subseteq B_n, \ (\text{if } n > 0),$$

and

$$h_n \in {}^{\mu}\nu_{\omega_1} \cap \bar{K}.$$

Note that $N^i \downarrow \nu_i = M \downarrow \nu_i = \bar{K} \downarrow \nu_i$, as on the one hand the iteration after index i is beyond ν_i and on the other hand $\bar{K} = K_\alpha^\sigma \subseteq M_\alpha^\sigma = M$.

Bob responds to Alice's move by playing a sequence $t^n = \langle t_i^n \mid i < \omega_1\rangle$ satisfying the following three conditions:

i) For each $i < \omega_1$, t_i^n maps a submodel of $N^i \downarrow \nu_i$ of size at most $\aleph_1$ elementarily into $N^{\omega_1} \downarrow \nu_{\omega_1}$, i. e., $t_i^n \ : \ N^i \downarrow \nu_i \hookrightarrow_{\Sigma_\omega} N^{\omega_1} \downarrow \nu_{\omega_1}$ and $\mathrm{card}(t_i^n) \leqslant \aleph_1$.

ii) If $n > 0$, then for each $i < \omega_1$, t_i^n extends t_i^{n-1}, i. e., $t_i^{n-1} \subseteq t_i^n$.

iii) For sufficiently large i, $\mathrm{rge}(t_i^n)$ "covers" $B_n \cap h_n{}'' \mu_i$, i. e., $\exists j < \omega_1 \, \forall i \in [j, \omega_1) \, \big(B_n \cap h_n{}'' \mu_i \subseteq \mathrm{rge}(t_i^n)\big)$.

If Bob is unable to find such a sequence, then Alice wins the game. Otherwise, i. e., if the number of moves is infinite, Bob wins. Thus Bob tries to construct, in the course of the game, an approximation to the direct limit system $\langle N^i \mid i < \omega_1 \rangle$, $\langle \pi_{ij} \mid i \leqslant j < \omega_1 \rangle$. Alice can require certain points (given by B_n and h_n) to be considered in this approximation, thus making the task harder for Bob.

Note that $\bar{H}$ is closed under ω_1-sequences and also that $N^i {\downarrow} \nu_i = \bar{K} {\downarrow} \nu_i \subseteq \bar{H}$, for any $i \leqslant \omega_1$. Thus any move that either Alice or Bob makes is an element of $\bar{H}$: they all have cardinality at most $\aleph_1$ and are subsets of $\bar{H}$. Again by ω_1-closure, the whole run of the game is then also an element of $\bar{H}$. Also note that the game $\mathfrak{G}$ is closed and hence determined. Thus either Alice or Bob must have a winning strategy. This last statement holds relativized to $\bar{H}$, too.

Claim 11 Bob has a winning strategy τ for $\mathfrak{G}$ in $\bar{H}$, i. e.,

$$\bar{H} \models \exists \tau \, \big(\tau \text{ is a winning strategy for Bob}\big).$$

Proof Suppose not. Then Alice has a winning strategy σ for $\mathfrak{G}$ in $\bar{H}$. Consider a run of the game where Alice plays $\langle B_n, h_n \rangle$ according to σ and where Bob responds with the following sequence t^n. For $i < \omega_1$, let H_i be a hull of $\pi_{i\omega_1}^{-1}{}'' B_n$ in $\bar{K} {\downarrow} \nu_i$, $\mathrm{card}(H_i) \leqslant \aleph_1$, constructed in V. Let $t_i^n := \pi_{i\omega_1} \restriction H_i$. Then by the above remark, $t^n = \langle t_i^n \mid i < \omega_1 \rangle \in \bar{H}$.

But these choices are always legal moves for Bob: Condition i) is satisfied by making $H_i \prec \bar{K} \downarrow \nu_i = N^i \downarrow \nu_i$. Since Alice always chooses sets B_n of size $\aleph_1$, $\mathrm{card}(H_i) \leqslant \aleph_1$ can also always be achieved. Condition ii) is fulfilled as Alice chooses $B_n \supseteq B_{n-1}$. Finally, note that the iteration from N^{ω_1} to $M \supseteq \bar{K}$ is beyond ν_{ω_1}. Thus if $h_n \in {}^{\mu}\nu_{\omega_1} \cap \bar{K}$, then $h_n \in N^{\omega_1}$, by [Koe89, Lemma 17.2 (iii)]. Subsequently, find some $j < \omega_1$ such that there exists $\bar{h} \in N^j$ satisfying $\pi_{j\omega_1}(\bar{h}) = h_n$. Now let $i \in [j, \omega_1)$ and set $\bar{h}_i := \pi_{ji}(\bar{h})$. For any $\xi \in B_n \cap h_n{}'' \mu_i$, let $\bar{\xi}_i < \mu_i$ be such that $h_n(\bar{\xi}_i) = \xi$. Then $\xi = h_n(\bar{\xi}_i) = \pi_{i\omega_1}(\bar{h}_i)(\bar{\xi}_i) = \pi_{i\omega_1}\left(\bar{h}_i(\bar{\xi}_i)\right) \in \mathrm{rge}(\pi_{i\omega_1})$. Thus $B_n \cap h_n{}'' \mu_i \subseteq \mathrm{rge}(\pi_{i\omega_1})$. Furthermore, $\xi \in B_n \cap \mathrm{rge}(\pi_{i\omega_1})$ implies that $\pi^{-1}(\xi) \in H_i =$ hull of

$\pi_{i\omega_1}^{-1}{}''B_n$, so that $\xi = t_i^n(\pi_{i\omega}^{-1}(\xi)) \in \mathrm{rge}(t_i^n)$. So for sufficiently large i, $\mathrm{rge}(t_i^n)$ indeed covers $B_n \cap h_n{}''\mu_i$, and condition iii) is also satisfied.

Thus this run of the game $\mathfrak{G}$ is infinite, so that Bob wins although Alice was using her winning strategy — contradiction. □(Claim 11)

Let $\tau \in \bar{H}$ be any winning strategy for Bob. Given $\beta < \nu_{\omega_1}$, we want to show that this will yield a candidate for a thread for β. Consider a run of the game $\mathfrak{G}$ where Bob plays according to τ and where Alice starts out by playing the singleton $B_0 := \{\beta\}$ together with $h_0 := c_\beta$, the function with constant value β (and domain μ, of course). Then, by clause iii) for Bob's moves, $\beta \in \mathrm{rge}(t_i^0)$ for sufficiently large i. Thus there is $j < \omega_1$ and there are β_i, for $i \in [j, \omega_1)$, such that $t_i^0(\beta_i) = \beta$. Choose some arbitrary β_i, for $i < j$, and let $\tau(\beta)$ denote the sequence $\langle \beta_i \mid i < \omega_1 \rangle$. Thus $\tau(\beta)$ is a thread for β in the sense of Bob's approximation t^0 to the real direct limit system. We will show next that in a certain sense the true thread for β is minimal among these approximations.

Consider the ordering of sequences of ordinals by eventual dominance:

7.10 Definition Given sequences $c = \langle \gamma_i \mid i < \omega_1 \rangle$ and $d = \langle \delta_i \mid i < \omega_1 \rangle$, let $c \preccurlyeq d$ (or $c \prec d$) iff for sufficiently large i, $\gamma_i \leqslant \delta_i$ (or $\gamma_i < \delta_i$), i. e., $\exists j < \omega_1\ \forall i \in [j, \omega_1)\ \Big(\gamma_i \leqslant \delta_i\Big)$ (or $\gamma_i < \delta_i$).

For $\beta < \nu_{\omega_1}$ let $\langle \check{\beta}_i \mid i < \omega_1 \rangle$ denote the (unique maximal) thread for β in N^{ω_1}.

Claim 12 Let τ be some winning strategy for Bob and let $\beta < \nu_{\omega_1}$. Then $\tau(\beta) \succcurlyeq \langle \check{\beta}_i \mid i < \omega_1 \rangle$.

Proof Let $\tau(\beta) = \langle \beta_i \mid i < \omega_1 \rangle$, where τ is some winning strategy for Bob. Assume for a contradiction that $\tau(\beta) \not\succcurlyeq \langle \check{\beta}_i \mid i < \omega_1 \rangle$, and consider the run of game $\mathfrak{G}$ where Alice starts out by playing $B_0 := \{\beta\}$ and $h_0 := c_\beta$. Bob responds by playing $t^n = \langle t_i^n \mid i < \omega_1 \rangle$ according to τ. For $n > 0$, Alice chooses B_n and h_n as follows:

$$B_n := \{\beta\} \cup \bigcup_{i < \omega_1} \pi_{i\omega_1}{}''\Big(\mathrm{dom}(t_i^{n-1}) \cap \mathrm{On}\Big).$$

Since all t_i^{n-1} have cardinality at most $\aleph_1$, so does B_n, so that $B_n \in [\nu_{\omega_1}]^{\leqslant\omega_1}$ as required. By induction it also follows that $B_n \supseteq B_{n-1}$, using the fact that t_i^{n-1} extends t_i^{n-2} (if $n > 1$). Note that by the ω_1-closure of $\bar{H}$ B_n is an element of $\bar{H}$. We now wish to apply the weak covering lemma 5.22 to choose h_n. $\mu = \mu_{\omega_1}$ is not overlapped in $\bar{K} = (\mathrm{K}^c)^{\bar{H}}$, as it is measurable and by $\neg\,\mathrm{L}^{\mathrm{strong}}$ there are no overlapping extendersequences. Also, $\nu_{\omega_1} = O^{\bar{K}}(\mu)$, since an end-extension of $\bar{K}$ is produced by iterating N^{ω_1} simply, beginning with $E^{N^{\omega_1}}_{\nu_{\omega_1}}$. $B_n \subseteq \nu_{\omega_1}$ and $(\mathrm{card}(B_n)^{\aleph_0} < \mathrm{card}(\mu))^{\bar{H}}$. Thus we can conclude that there is some $h \in \bar{K}$, $h : \kappa \to \nu_{\omega_1}$ such that $B_n \subseteq \mathrm{rge}(h_n)$.

This effectively forces Bob to respect all of B_n with his next move, for sufficiently large i. This would not be a priori impossible, were it not for our assumption that $\tau(\beta) \not\leqslant^* \langle\check{\beta}_i \mid i < \omega_1\rangle$, i. e., that the true thread for β unboundedly often lies strictly above Bob's approximation to it. This latter fact is now used to show that Bob cannot win the game:

Suppose to the contrary that Bob does win, i. e., that the number of moves is infinite. Let $n \in \omega$. Then $h_n \in {}^{\mu}\nu_{\omega_1} \cap \bar{K} \subseteq M$. M is a simple fine iterate of N^{ω_1} beyond ν_{ω_1}, so that we can use [Koe89, Lemma 17.2 iii)] again to conclude that $h_n \in N^{\omega_1}$. Thus there is some $j < \omega_1$ and some $\bar{h} \in N^j$ such that $h_n = \pi_{j\omega_1}(\bar{h})$. For any $i \in [j, \omega_1)$, set $\bar{h}_i := \pi_{ji}(\bar{h})$. We claim that

$$B_n \cap \mathrm{rge}(\pi_{i\omega_1}) \subseteq h_n{}''\mu_i.$$

To see this, let $\xi \in B_n \cap \mathrm{rge}(\pi_{i\omega_1})$, say $\xi = \pi_{i\omega_1}(\bar{\xi})$. By construction $B_n \subseteq \mathrm{rge}(h_n)$, so that

$$N^{\omega_1} \models \exists\xi' < \mu\ \Big(h_n(\xi') = \xi\Big).$$

By elementarity,

$$N^{i} \models \exists\xi' < \mu_i\ \Big(\bar{h}_i(\xi') = \bar{\xi}\Big).$$

So pick some $\xi' < \mu_i$ such that

$$N^i \models \bar{h}_i(\xi') = \bar{\xi}.$$

Since the iteration from N^i to N^{ω_1} is normal and above μ_i

$$N^{\omega_1} \models h_n(\xi') = \xi,$$

i. e., $\xi \in h_n{}''\mu_i$, as claimed.

By condition iii) for Bob's moves, there is $j_n < \omega_1$ such that

$$\forall i \in [j_n, \omega_1) \ \Big(B_n \cap \mathrm{rge}(\pi_{i\omega_1}) \subseteq B_n \cap h_n{}''\mu_i \subseteq \mathrm{rge}(t_i^n)\Big).$$

Since this is true for all $n \in \omega$, let $j := \sup\{j_n \mid n \in \omega\} < \omega_1$. Then

$$\forall n \in \omega \ \forall i \in [j, \omega_1) \ \Big(B_n \cap \mathrm{rge}(\pi_{i\omega_1}) \subseteq \mathrm{rge}(t_i^n)\Big). \tag{1}$$

W.l.o.g. pick j large enough so that also

$$\forall i \in [j, \omega_1) \ \Big(t_i^0(\beta_i) = \beta\Big). \tag{2}$$

This is possible by definition of $\langle \beta_i \mid i < \omega_1 \rangle$ as $\tau(\beta)$. (Recall that Alice started out by playing $B_0 = \{\beta\}$ and Bob answered according to τ.) Finally, we want to make use of our assumption that $\tau(\beta) \not\geq \langle \check{\beta}_i \mid i < \omega_1 \rangle$. This ensures that for some $i > j$ $\beta_i < \check{\beta}_i$, i. e.,

$$\exists i \in [j, \omega_1) \ \Big(\pi_{i\omega_1}(\beta_i) < \pi_{i\omega_1}(\check{\beta}_i) = \beta\Big). \tag{3}$$

We can now inductively go back and forth between N^i and N^{ω_1} and construct a descending sequence of ordinals, as shown in Figure 7.7.

Fix some i as in equation (3). Let $\eta_0 := \pi_{i\omega_1}(\beta_i) < \beta$. Note that $\eta_0 \in B_1 \cap \mathrm{rge}(\pi_{i\omega_1})$: By equation (2), $\beta_i \in \mathrm{dom}(t_1^0)$, so that $\eta_0 = \pi_{i\omega_1}(\beta_i) \in \pi_{i\omega_1}{}''(\mathrm{dom}(t_1^0) \cap \mathrm{On}) \subseteq B_1$. Assume that η_{n-1} had been defined and shown to be an element of $B_n \cap \mathrm{rge}(\pi_{i\omega_1})$. Then set

$$\eta_n := \pi_{i\omega_1}\Big(\big(t_i^n\big)^{-1}(\eta_{n-1})\Big).$$

This is well-defined, as by equation (1) we have that $B_n \cap \mathrm{rge}(\pi_{i\omega_1}) \subseteq \mathrm{rge}(t_i^n)$, so that $\zeta_n := \big(t_i^n\big)^{-1}(\eta_{n-1})$ is defined. Also $\eta_n = \pi_{i\omega_1}(\zeta_n) \in \pi_{i\omega_1}{}''(\mathrm{dom}(t_i^n) \cap \mathrm{On}) \subseteq B_{n+1}$, by the definition of B_{n+1}. Thus the inductive definition goes through.

We claim that $\forall n \in \omega \ \Big(\eta_{n+1} < \eta_n\Big)$. First, note that $\eta_0 = \pi_{i\omega_1}(\beta_i) < \beta = t_i^0(\beta_i)$, so that $\big(t_1^0\big)^{-1}(\eta_0) < \beta_i$. This implies that

$$\eta_1 = \pi_{i\omega_1}\Big(\big(t_i^0\big)^{-1}(\eta_0)\Big) < \pi_{i\omega_1}(\beta_i) = \eta_0.$$

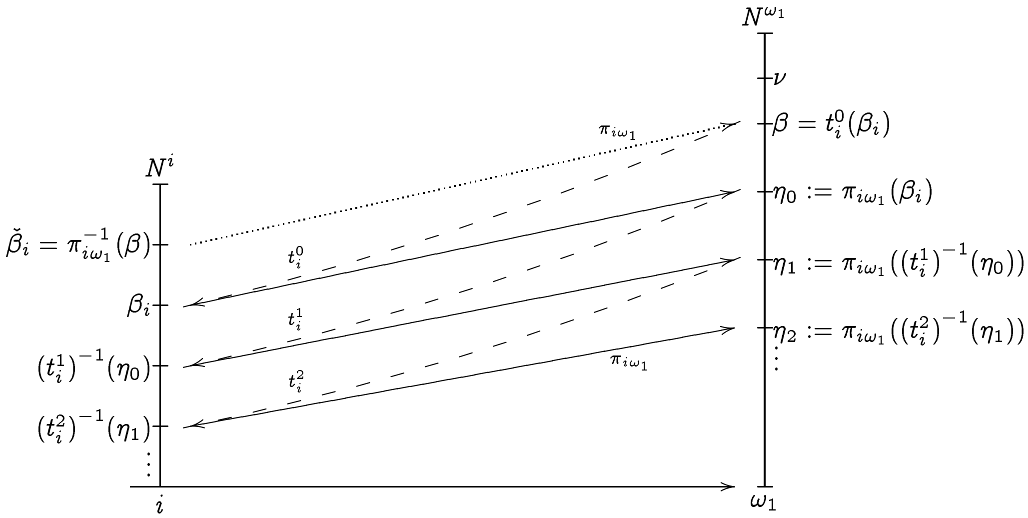

Figure 7.7: To construct an infinite descending sequence of ordinals, start with β in N^{ω_1}, take its pre-image under t_i^0 to go to N^i, then use $\pi_{i\omega_1}$ to return to N^{ω_1} and start over again, using t_i^1 next.

Inductively, if $\eta_n < \eta_{n-1}$, $n > 0$, then

$$\begin{aligned}
\eta_{n+1} &= \pi_{i\omega_1}\Big(\big(t_i^{n+1}\big)^{-1}(\eta_n)\Big) \\
&< \pi_{i\omega_1}\Big(\big(t_i^{n+1}\big)^{-1}(\eta_{n-1})\Big), && \text{since } \eta_{n-1} \in \mathrm{rge}(t_i^n) \subseteq \mathrm{rge}(t_i^{n+1}), \\
&= \pi_{i\omega_1}\Big(\big(t_i^{n}\big)^{-1}(\eta_{n-1})\Big), && \text{since } t_i^n \subseteq t_i^{n+1}, \\
&= \eta_n && \text{by the definition of } \eta_n.
\end{aligned}$$

This is a contradiction. Thus Alice wins the game. But Bob was playing according to a winning strategy τ. Contradiction yet again! Thus our initial assumption that $\tau(\beta) \not\preccurlyeq \langle \check{\beta}_i \mid i < \omega_1 \rangle$ must have been wrong, and the claim is proved. $\square$(Claim 12)

Now the argument for Claim 11 shows that, for any $\beta < \nu_{\omega_1}$, Bob has a winning strategy $\tau \in \bar{H}$ such that $\tau(\beta)$ is a thread for β in N^{ω_1}. Thus if we let

$$b_\beta := \text{the } \preccurlyeq\text{-minimum of } \{\tau(\beta) \mid \tau \text{ is a winning strategy for Bob}\},$$

then b_β is a thread for β in N^{ω_1}, defined inside $\bar{H}$: The set of possible threads $\tau(\beta)$ consists only of threads greater or equal than the canonical thread for β, by Claim 12, and it contains one which is eventually equal to this thread. But this proves Claim 9, and with it the theorem. $\square$(Claim 9)

$\blacksquare$

As we already noted at the end of Chapter 6, our original goal was to prove that the existence of an irregular ultrafilter entailed the existence of an inner model for a strong cardinal. Under the present circumstances, the Gitik Game is not strictly necessary: all measures involved are ultrafilters, no recourse to extenders is needed, and ultrafilters are reconstructible in an ω-closed model after an ω-long iteration, just from their Prikry-sequence. However, we considered the Gitik Game interesting enough to merit a detailed exposition, and it also serves to locate the exact point where the proof breaks down for higher orders of measurability, namely Lemma 6.2.

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